An energy constrained method for the existence of layered type solutions of NLS equations.

Francesca Alessio[1] and Piero Montecchiari[2]

Dipartimento di Ingegneria Industriale e Scienze Matematiche,
Università Politecnica delle Marche, Via Brecce Bianche – I 60131 Ancona,
e-mail: [1]f.g.alessio@univpm.it, [2]p.montecchiari@univpm.it.

Abstract. We study the existence of positive solutions on $\mathbb{R}^{N+1}$ to semilinear elliptic equation $-\Delta u+u=f(u)$ where $N\geq 1$ and f is modeled on the power case $f(u)=|u|^{p-1}u$. Denoting with c the mountain pass level of $V(u)=\frac{1}{2}\|u\|^2_{H^1(\mathbb{R}^N)}-\int_{\mathbb{R}^N}F(u)\,dx$, $u\in H^1(\mathbb{R}^N)$ ($F(s)=\int_0^s f(t)\,dt$), we show, via a new energy constrained variational argument, that for any $b\in[0,c)$ there exists a positive bounded solution $v_b\in C^2(\mathbb{R}^{N+1})$ such that $E_{v_b}(y)=\frac{1}{2}\|\partial_y v_b(\cdot,y)\|^2_{L^2(\mathbb{R}^N)}-V(v_b(\cdot,y))=-b$ and $v(x,y)\to 0$ as $|x|\to+\infty$ uniformly with respect to $y\in\mathbb{R}$. We also characterize the monotonicity, symmetry and periodicity properties of v_b.

Key Words. Semilinear elliptic equations, locally compact case, variational methods, Energy constraints.

Mathematics Subject Classification: 35J60, 35B08, 35B40, 35J20, 34C37.

[1,2] Partially supported by the PRIN2009 grant "Critical Point Theory and Perturbative Methods for Nonlinear Differential Equations"

1 Introduction

In this paper we study the existence of positive solutions on $\mathbb{R}^{N+1}$ to semilinear elliptic equations

$$-\Delta u + u = f(u) \tag{E}$$

where $N \geq 1$ and f is a nonlinearity which can be thought modeled on the power case $f(u) = |u|^{p-1}u$ with p subcritical and greater than 1. Equations of this kind are used in various fields of Physics such as, for example, plasma or laser self-focusing models (see [27] and the references therein). They arise in particular in the study of standing waves (stationary states) solutions of the corresponding nonlinear Schrödinger type equations.

Starting with the work by W. A. Strauss, [28], the problem of finding and characterizing positive solutions $v \in H^1(\mathbb{R}^{N+1})$ of (E) has been widely studied. We refer to the paper by H. Berestycki and P.L. Lions [7] (in the case $N \geq 2$, see [8] for $N = 1$) where nearly optimal existence results regarding least energy solutions for (E) are obtained. Their mountain pass characterization, and so information about their Morse index, is given by L. Jeanjean and K. Tanaka in [16]. In the pure power case, uniqueness and non degeneracy properties of solutions of (E) in $H^1(\mathbb{R}^{N+1})$ was derived by M.K. Kwong in [17]. Regarding the uniqueness problem for more general nonlinearity f, we refer to the paper by J. Serrin and M. Tang, [26], and to the references therein.

A new kind of entire solutions of (E) has been introduced by N. Dancer in [12]. Denoting $(x, y) \in \mathbb{R}^N \times \mathbb{R}$ a point in $\mathbb{R}^{N+1}$, we note that a ground state solution $u_0(x)$ of (E) in $\mathbb{R}^N$ can be thought as a solution of (E) on $\mathbb{R}^{N+1}$, which is constant with respect to the y variable. In the pure power case (or anyhow assuming the nondegeneracy of the ground state solution) Dancer proved, by using bifurcation and continuation arguments, the existence of a continuous branch of entire positive solutions of (E) in $\mathbb{R}^{N+1}$ bifurcating from the *cylindric type* solution u_0. These solutions are periodic in the variable y and decay to zero as $|x| \to +\infty$. Different periodic Dancer's solutions (suitably rotated) were then used in the pure power case as prescribed asymptotes in the constructions of *multiple ends* solutions of (E) by A. Malchiodi in [20] and by M. del Pino, M. Kowalczyk, F, Pacard and J. Wei in [13].

Related to the above papers is the one by C. Gui, A. Malchiodi and H. Xu, [15], where qualitative properties (such as radial symmetry with respect to the variable x and eveness with respect to y) of positive solutions $v(x, y)$ of (E) which decay to zero as $|x| \to +\infty$ (uniformly w.r.t. y) are established. Their study is based on moving plane techniques together with the use of some Hamiltonian identities which are connected with the Lagrangian structure of that kind of problem.

To describe the Hamiltonian identities which are used in [15] and to introduce precisely the problem studied in the present paper, note that prescribing the decay properties of a solution v only with respect to the variable $x \in \mathbb{R}^N$, naturally gives to the variable y the role of an evolution variable. In this respect, as usual in the evolution problems, all the solutions v of (E) described above belong to the space $X = L^2_{loc}(\mathbb{R}, H^1(\mathbb{R}^N)) \cap H^1_{loc}(\mathbb{R}, L^2(\mathbb{R}^N))$ and verify (at least in a weak sense) the evolution equation

$$\partial_y^2 v(\cdot, y) = V'(v(\cdot, y)), \quad y \in \mathbb{R}, \tag{1.1}$$

where V' is the gradient in $H^1(\mathbb{R}^N)$ of the Euler functional relative to equation (E) on $\mathbb{R}^N$,

$$V(u) = \int_{\mathbb{R}^N} \tfrac{1}{2}|\nabla u|^2 + \tfrac{1}{2}|u|^2 - F(u)\, dx, \quad u \in H^1(\mathbb{R}^N),$$

where $F(s) = \int_0^s f(t)\, dt$. We will refer to this kind of solutions as *layered solutions* of (E).

Noting that equation (1.1) has Lagrangian structure, one can think to the variable y as a *time* variable and to the functional $U = -V$ as the *energy potential* of the infinite dimensional dynamical system. Every layered solution v defines a trajectory $y \in \mathbb{R} \to v(\cdot, y) \in H^1(\mathbb{R}^N)$, solution to (1.1). In this connection, any $u \in H^1(\mathbb{R}^N)$ which solves (E) is an equilibrium of (1.1) and the solutions found by Dancer are periodic orbits of the system. Since the system is autonomous, if v is a layered solution to (E) then the *Energy* function

$$y \to E_v(y) = \frac{1}{2}\|\partial_y v(\cdot, y)\|^2_{L^2(\mathbb{R}^N)} - V(v(\cdot, y))$$

is constant (a formal proof of this Hamiltonian identity for a general class of elliptic equations can be found in [9] and [14], see also [2] for the case of Allen Cahn equations).

In the present paper, in analogy with the study already done for Allen Cahn type equation in [2], [3], [4] (see also [1] for Allen Cahn system of equations), we study the problem of finding layered solution of (E) with prescribed energy. In particular we study the problem of looking for *connecting orbit* solutions with prescribed energy.

To be more detailed, we precise our assumption on the non linearity f. We assume that

$(f1)$ $f \in C^1(\mathbb{R})$,

$(f2)$ there exists $C > 0$ and $p \in (1, 1 + \frac{4}{N})$ such that $|f(t)| \leq C(1 + |t|^p)$ for any $t \in \mathbb{R}$,

$(f3)$ there exists $\mu > 2$ such that $0 < \mu F(t) \leq f(t)t$ for any $t \neq 0$, where $F(t) = \int_0^t f(s)\, ds$,

$(f4)$ $f(t)t < f'(t)t^2$ for any $t \neq 0$.

As it is well known, $(f1)$–$(f4)$ are more than sufficient to guaranty that $V \in C^1(H^1(\mathbb{R}^N))$ and that it satisfies the geometrical assumptions of the Mountain pass Theorem. Setting $c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} V(\gamma(t))$, where $\Gamma = \{\gamma \in C([0, 1], H^1(\mathbb{R}^N)) \,|\, \gamma(0) = 0,\, V(\gamma(1)) < 0\}$, we have that $c > 0$ is an asymptotical critical level for V. Concentration compactness arguments allow to prove that c is actually the lowest positive critical level of V. Then, the definition of the Mountain pass level implies that given any $b \in [0, c)$ the sublevel $\{V \leq b\}$ is the union of two disjoint path connected sets $\mathcal{V}^b_-$ and $\mathcal{V}^b_+$, where we denote with $\mathcal{V}^b_-$ the one which contains 0. The main result of the present paper establishes that given any $b \in [0, c)$ there exists a layered solution v of (E) with $E_v = -b$ and which connects the set $\mathcal{V}^b_-$ and $\mathcal{V}^b_+$, in the sense that $\liminf_{y \to \pm\infty} \operatorname{dist}_{L^2(\mathbb{R}^N)}(v(\cdot, y), \mathcal{V}^b_\pm) = 0$. Precisely we prove that

Theorem 1.1 *If F satisfies $(f1) - (f4)$ then for any $b \in [0, c)$ the equation (E) has a solution $v_b \in C^2(\mathbb{R}^{N+1})$ with energy $E_{v_b} = -b$ and such that*

i) $v_b > 0$ on $\mathbb{R}^{n+1}$,

ii) $v_b(x, y) = v_b(|x|, y) \to 0$ as $|x| \to +\infty$, uniformly w.r.t. $y \in \mathbb{R}$,

iii) $\partial_r v_b(x, y) < 0$ for $r = |x| > 0$ and $y \in \mathbb{R}$.

Moreover, if $b > 0$,

iv) there exists $T_b > 0$ such that v_b is periodic of period $2T_b$ in the variable y and symmetric with respect to $y = 0$ and $y = T_b$.

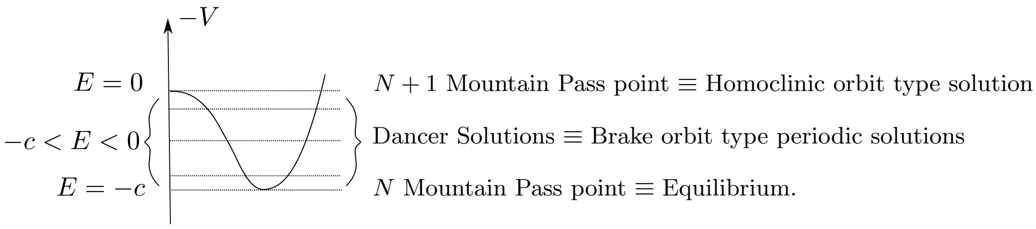
v) $\partial_y v_b(x, y) > 0$ on $\mathbb{R}^N \times (0, T_b)$, $v_b(\cdot, 0) \in \mathcal{V}^b_-$, $v_b(\cdot, T_b) \in \mathcal{V}^b_+$.

Finally, if $b = 0$,

v) $v_0 \in H^1(\mathbb{R}^{N+1})$ is radially symmetric and $\partial_r v_0 < 0$ for $r = |(x,y)| > 0$,

vi) $v_0(\cdot, 0) \in \mathcal{V}_+^0$ and v_0 is a mountain pass point of the Euler funcional relative to (E) on $H^1(\mathbb{R}^{N+1})$.

Theorem 1.1 gives the existence for any $b \in [0, c)$ of a positive layered solution v_b to (E) with energy $-b$ which is radially symmetric and decaying to 0 as $|x| \to +\infty$ uniformly with respect to $y \in \mathbb{R}$. When $b > 0$ the solution v_b is a *periodic solution* of period $2T_b$ which is symmetric with respect to $y = 0$ and $y = T_b$. It can be thought as a trajectory which oscillates back and forth along a simple curve connecting the two turning points $v_b(\cdot, 0) \in \mathcal{V}_-^b$ and $v_b(\cdot, T_b) \in \mathcal{V}_+^b$. These solutions, which we call *brake orbit type solutions*, are clearly analogous to the Dancer solutions. When $b = 0$ the solution v_0 defines a trajectory which emanates from $0 \in H^1(\mathbb{R}^N)$ as $y \to -\infty$, reaches the point $v(\cdot, 0) \in \mathcal{V}_+^0$ and goes back symmetrically to 0 for $y > 0$. It can been thought as a *homoclinic solution* to $0 \in H^1(\mathbb{R}^N)$ and it is in fact the mountain pass point of the Euler funcional relative to (E) on $H^1(\mathbb{R}^{N+1})$. Finally we can think at the mountain pass point of V in $H^1(\mathbb{R}^N)$ as an equilibrium of (1.1) at energy $-c$. The Energy diagram here below wants to summarize these considerations.



To prove Theorem 1.1 we make use of variational methods and we apply an Energy constrained variational argument already introduced and used in [2], [3] and [4]. Given $b \in [0, c)$, we look for minima of the renormalized functional

$$\varphi(v) = \int_{\mathbb{R}} \tfrac{1}{2} \|\partial_y v(\cdot, y)\|^2_{L^2(\mathbb{R}^N)} + (V(v(\cdot, y)) - b)\, dy$$

on the space of function $v \in X$ which are radially symmetric with respect to $x \in \mathbb{R}^N$, monotone decreasing with respect to $|x|$ and which verify

$$\liminf_{y \to \pm\infty} \mathrm{dist}_{L^2(\mathbb{R}^N)}(v(\cdot, y), \mathcal{V}_\pm^b) = 0 \text{ and } \inf_{y \in \mathbb{R}} V(v(\cdot, y)) \geq b. \tag{1.2}$$

Thanks to the constraint $\inf_{y \in \mathbb{R}} V(v(\cdot, y)) \geq b$, the functional φ is well defined on this class of functions. Moreover, its minimizing sequences admits limit points $\bar{v} \in X$ (a priori not verifying (1.2)) with respect to the weak topology of $H^1_{loc}(\mathbb{R}^2)$.

Defining $\bar{\sigma} = \sup\{y \in \mathbb{R} \,/\, v(\cdot, y) \in \mathcal{V}_-^b\}$ and $\bar{\tau} = \inf\{y > \bar{\sigma} \,/\, v(\cdot, y) \in \mathcal{V}_-^b\}$, we can prove that $-\infty \leq \bar{\sigma} < \bar{\tau} < +\infty$ (indeed $\bar{\sigma} > -\infty$ when $b > 0$) and $\lim_{y \to \bar{\sigma}^+} \mathrm{dist}(\bar{v}(\cdot, y), \mathcal{V}_-^b) = 0$, $\bar{v}(\cdot, \bar{\tau}) \in \mathcal{V}_+^b$ and $V(v(\cdot, y)) > 0$ for any $y \in (\bar{\sigma}, \bar{\tau})$. Then, the minimality properties of $\bar{v}$ allow us to prove that $\bar{v}$ solves in a classical sense the equation (E) on $\mathbb{R}^N \times (\bar{\sigma}, \bar{\tau})$ and $E_{\bar{v}}(y) = -b$ for any $y \in (\bar{\sigma}, \bar{\tau})$. This will imply that $\bar{v}$ satisfies the boundary conditions $\lim_{y \to \bar{\sigma}^+} \partial_y \bar{v}(\cdot, y) = \lim_{y \to \bar{\tau}^-} \partial_y \bar{v}(\cdot, y) = 0$ in L^2 and the entire solution v_b is recovered from $\bar{v}$ by translations, reflections and, eventually, periodic continuations.

The variational approach that we used is similar to the one already applyed in the study of the Allen Cahn type equation in [2], [3],[4], but the present case is much more complicated

due to some natural lack of compactness and weak semicontinuity of the problem. This mainly depends on the competition between the term $\|u\|^2_{H^1(\mathbb{R}^N)}$ and $\int_{\mathbb{R}^N} F(u)$ which enters in the definition of the potential functional $V(u)$ with different sign. This explain why we assume in $(f2)$ that $p < 1+4/N$. The exponent $p = 1+4/N$ is in fact critical with respect to the existence for the minimum problem $\inf\{V(u)\,|\,u \in H^1(\mathbb{R}^N),\ \|u\|_{L^2(\mathbb{R}^N)} = 1\}$ and, related to that, with respect to the property of orbital stability of the solutions of (E) in $H^1(\mathbb{R}^N)$. We recall that the ground state solutions of (E) in $H^1(\mathbb{R}^N)$ are stable when $1 < p < 1+4/N$ (see [11]) while the solutions of (E) in $H^1(\mathbb{R}^N)$ are unstable when $1 + 4/N \le p$ (see [6] for the case $1 + 4/N < p$ and [29] for $p = 1 + 4/N$). Another (related) criticality of the exponent $p = 1+4/N$ is the fact that the sets $\mathcal{V}^b_\pm$ have positive $L^2(\mathbb{R}^N)$ distance if and only if $p < 1 + 4/N$ (one can simply verify it by using dilations in the pure power case). Even if we think that this assumption is only technical and can be probably overcome, here we begin to study the problem in this more compact setting.

The paper is organized as follows. In section 2 we recall some properties of the functional V studying in particular the structure of the sublevel set $\mathcal{V}^b_\pm$. The study of the functional φ and the use of the energy constraint variational principle described above is contained in section 3.

Acknowledgments. We wish to thanks Andrea Malchiodi, Margherita Nolasco and Vittorio Coti Zelati for the useful comments and discussions.

Remark 1.1 Since we look for positive solution of (E) it is not restrictive to assume, and we will do it along the paper, that f is an odd function

$(f5)$ $f(t) = -f(-t)$ for any $t > 0$.

Moreover, we list also some plain consequences of $(f1)$-$(f4)$.

i) By $(f1)$ and $(f3)$ it is straightforward to verify that $f(0) = f'(0) = 0$ and so $f(t) = o(t)$ as $t \to 0$.

ii) By (i) and $(f2)$ we have

$$\forall \varepsilon > 0,\ \exists A_\varepsilon > 0 \text{ such that } |f(t)| \le \varepsilon|t| + A_\varepsilon |t|^p,\ \forall\, t \in \mathbb{R}, \tag{1.3}$$

from which we also derive

$$\forall \varepsilon > 0,\ \exists A_\varepsilon > 0 \text{ such that } |F(t)| \le \tfrac{\varepsilon}{2}|t|^2 + \tfrac{A_\varepsilon}{p+1}|t|^{p+1},\ \forall\, t \in \mathbb{R}. \tag{1.4}$$

iii) By $(f3)$, if $t \neq 0$ and $s > 0$, we have $\frac{d}{ds}F(st) = \frac{1}{s}f(st)st > \frac{\mu}{s}F(st)$. Hence,

$$F(st) > F(t)s^\mu \text{ whenever } t \neq 0 \text{ and } s > 1. \tag{1.5}$$

iv) By $(f4)$, one plainly verify that, for any $t \neq 0$,

$$\text{the function } s \mapsto \tfrac{1}{s}f(st)t \text{ is strictly increasing for } s > 0. \tag{1.6}$$

For the sake of brevity in the notation, along the paper we denote $\|u\| \equiv \|u\|_{H^1(\mathbb{R}^N)}$, $\|u\|_p = \|u\|_{L^p(\mathbb{R}^N)}$ and $\langle u, v\rangle = \langle u, v\rangle_{H^1(\mathbb{R}^n)}$, $\langle u, v\rangle_2 = \langle u, v\rangle_{L^2(\mathbb{R}^n)}$ for $n = N$ or $n = N+1$. Moreover $\mathrm{dist}(A, B) \equiv \mathrm{dist}_{L^2(\mathbb{R}^N)}(A, B) = \inf_{v\in A,\, w\in B} \|v - w\|_2$ and $\mathrm{dist}(u, B) \equiv \inf_{v\in B} \|u - v\|_2$ for $A, B \subset L^2(\mathbb{R}^N)$, $u \in L^2(\mathbb{R}^N)$. Given $y \in \mathbb{R}^N$ we set $B_r(y) \equiv \{x \in \mathbb{R}^N \,/\, |x| < r\}$ and $B_r \equiv B_r(0)$.

2 The Potential functional

In this chapter, we study some properties of the functional $V : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$V(u) = \tfrac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(u(x))\,dx. \tag{2.1}$$

2.1 The Mountain Pass structure.

Here we list some classical properties of V, in particular the ones regarding its mountain pass behaviour.
First of all we recall that V is regular on $H^1(\mathbb{R}^N)$ (see e.g [5] and [21]).

Lemma 2.1 *$V \in \mathcal{C}^2(H^1(\mathbb{R}^N))$ with $V'(u)h = \int_{\mathbb{R}^N} \nabla u\,\nabla h + u\,h - f(u)h\,dx$ and $V''(u)h\cdot h = \int_{\mathbb{R}} |\nabla h|^2 + |h|^2 - f'(u)h^2\,dx$ for all $h \in H^1(\mathbb{R}^N)$.*

Moreover the functional V satisfies the *geometrical hypotheses* of the Mountain Pass Theorem. Indeed, since $p+1 < 2^*_N$, by the Sobolev Immersion Theorem and Remark 1.1-(ii) we obtain

Lemma 2.2 *There exists $\rho \in (0,1)$ such that if $u \in H^1(\mathbb{R}^N)$ satisfies $\|u\| \le \rho$ then $V(u) \ge \frac{1}{4}\|u\|^2$ and $V'(u)v \ge \langle u, v\rangle - \frac{1}{2}\|u\|\|v\|$ for all $v \in H^1(\mathbb{R}^N)$.*

By Lemma 2.2 in particular we get that $\inf_{\|u\|=\rho} V(u) \ge \frac{1}{4}\rho^2 > 0$. Moreover, by Remark 1.1-(iii), we have that if $u \in H^1(\mathbb{R}^N)\setminus\{0\}$ and $s > 1$ then $V(su) = s^2\|u\|^2 - \int_{R^n} F(su)\,dx \le s^2\|u\|^2 - s^\mu \int_{R^n} F(u)\,dx$ and hence $V(su) \to -\infty$ as $s \to +\infty$ for all $u \in H^1(\mathbb{R}^N)\setminus\{0\}$. Hence, defining

$$\Gamma = \{\, \gamma \in C([0,1], H^1(\mathbb{R}^N)) \;:\; \gamma(0) = 0\,,\ \gamma(1) \ne 0 \text{ and } V(\gamma(1)) \le 0\,\}$$

and setting

$$c = \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} V(\gamma(s)),$$

we get $c \ge \frac{1}{4}\rho^2$ and by the Mountain Pass Theorem (see e.g. [25]) there exists a Palais Smale sequence for V at level c.
Moreover, by $(f3)$, the following inequality holds true

$$\mu V(u) - V'(u)u = (\tfrac{\mu}{2} - 1)\|u\|^2 + \int_{\mathbb{R}^N} f(u)u - \mu F(u) \ge \tfrac{\mu-2}{2}\|u\|^2, \tag{2.2}$$

from which in particular we derive that the Palais Smale sequences of V are bounded in $H^1(\mathbb{R}^N)$. By (2.2), we obtain also that if $V'(u) = 0$ and $u \ne 0$ then $V(u) \ge \frac{\mu-2}{2\mu}\|u\|^2$, showing that V has not critical points (or Palais Smale sequences) at negative levels.

The existence of a mountain pass critical point of V can then be deduced by using concentration compactness argument. We have

Proposition 2.1 *There exists $w_0 \in H^1(\mathbb{R}^N)$ such that $V(w_0) = c$ and $V'(w_0) = 0$. Moreover $w_0 \in C^2(\mathbb{R}^N)$ is a solution of (E) on $\mathbb{R}^N$, $w_0 > 0$, $w_0(x) \to 0$ as $|x| \to +\infty$ and, up to translations, w_0 is radially symmetric about the origin with $\partial_r w_0 < 0$ for $r = |x| > 0$.*

We refer for a proof to [7], for $N \ge 3$ and [8] for $N = 2$, where a more general existence results regarding least energy solutions for scalar field equations is given. Their Mountain pass characterization is proved in [16]. The case $N = 1$ is easier and we omit the proof.

Fixed $u \in H^1(\mathbb{R}^N)$, the assumption $(f4)$ allows us to describe the behaviour of V along the rays $\{tu \,|\, t \ge 0\}$ in $H^1(\mathbb{R}^N)$.

Lemma 2.3 *For every $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ there exists $t_u > 0$ such that*

$$\tfrac{d}{dt} V(tu) > 0 \text{ for } t \in (0, t_u) \text{ and } \tfrac{d}{dt} V(tu) < 0 \text{ for } t \in (t_u, +\infty). \tag{2.3}$$

Moreover $V(t_u u) \geq c$ and for any $b \in (0, c)$ there exist unique $\alpha_{u,b} \in (0, t_u)$ and $\omega_{u,b} \in (t_u, +\infty)$ such that $V(\alpha_{u,b} u) = V(\omega_{u,b} u) = b$. Finally the function $t \mapsto V'(tu)tu$ is decreasing in $(t_u, +\infty)$.

Proof. We have

$$\tfrac{d}{dt} V(tu) = V'(tu)u = t(\|u\|^2 - \tfrac{1}{t} \int_{\mathbb{R}^N} f(tu)u \, dx). \tag{2.4}$$

By $(f4)$ the function $t \mapsto \frac{1}{t} \int_{\mathbb{R}^N} f(tu)u \, dx$ is stricly increasing in $(0, +\infty)$ for any $u \neq 0$ and so, by (2.4), the fuction $\frac{d}{dt} V(tu)$ can change sign at most in one point $t_u > 0$. Then (2.3) follows since $V(0) = 0$, $V(su) \geq \frac{1}{4} s^2 \|u\|^2$ for $s \in (0, \rho/\|u\|)$ and $V(su) \to -\infty$ as $s \to +\infty$. By (2.3) we deduce $V(t_u u) = \max_{s \geq 0} V(su)$, and, by the definition of the mountain pass level, we have $V(t_u u) \geq c$. Given $b \in [0, c)$, since $V(0) = 0$, $V(tu) < 0$ for t large and $V(t_u u) \geq c$, by continuity there exist (unique by (2.3)) $0 \leq \alpha_{u,b} < t_u < \omega_{u,b}$ such that $V(\alpha_{u,b} u) = V(\omega_{u,b} u) = b$. We finally note that by $(f4)$ we have $\frac{d^2}{dt^2} V(tu) = \|u\|^2 - \int_{\mathbb{R}^N} f'(tu)u^2 \, dx \leq \|u\|^2 - \frac{1}{t} \int_{\mathbb{R}^N} f(tu)u \, dx < 0$ for any $t > t_u$. We conclude that $\frac{d}{dt} V'(tu)tu = \frac{d}{dt}(t \frac{d}{dt} V(tu)) = t \frac{d^2}{dt^2} V(tu) + \frac{d}{dt} V(tu) < 0$ for any $t > t_u$. $\square$

Remark 2.1 Note that if $V'(u)u = 0$ and $u \neq 0$ we have $\frac{d}{dt} V(tu)\big|_{t=1} = V'(u)u = 0$ and so $t_u = 1$. Then, by Lemma 2.3, $V(u) = V(t_u u) \geq c$ whenever $u \neq 0$ and $V'(u)u = 0$.

Remark 2.2 We note that, since by $(f2)$ we have $p < 2^*_{N+1} - 1$, all the results stated and proved in the present sections holds unchanged for all $m \in \{1, \dots, N+1\}$ considering the functionals

$$V_m(u) = \tfrac{1}{2} \|u\|^2_{H^1(\mathbb{R}^m)} - \int_{\mathbb{R}^m} F(u(x)) \, dx, \quad u \in H^1(\mathbb{R}^m).$$

In particular, denoting c_m the mountain pass level of V_m in $H^1(\mathbb{R}^m)$, Proposition 2.1 establishes that V_m has a positive, radially symmetric, critical point $w \in H^1(\mathbb{R}^m)$ at the level c_m.

2.2 Further properties of V on the space of radial functions. The sublevels $\mathcal{V}^b_-$ and $\mathcal{V}^b_+$

From now on we reduce ourself to work on the subspace of H^1 constituted by radial functions: $H^1_r(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N) \,/\, u(x) = u(|x|)\}$. We recall that by the Strauss Lemma (see [28], [19]) $H^1_r(\mathbb{R}^N)$ is compactly embedded in $L^q(\mathbb{R}^N)$ for all $q \in (2, 2^*_N)$. Thanks to the Strauss Lemma the functional V is weakly lower semicontinuous on $H^1_r(\mathbb{R}^N)$.

Lemma 2.4 *Let $u_n \to u$ and $v_n \to v$ weakly in $H^1_r(\mathbb{R}^N)$. Then*

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} F(u_n) dx = \int_{\mathbb{R}^N} F(u) dx \quad \text{and} \quad \lim_{n \to +\infty} \int_{\mathbb{R}^N} f(u_n) v_n \, dx = \int_{\mathbb{R}^N} f(u) v dx.$$

Hence $V(u) \leq \liminf_{n \to +\infty} V(u_n)$, $V'(u)u \leq \liminf_{n \to +\infty} V'(u_n)u_n$ and, for every $h \in H^1_r(\mathbb{R}^N)$, $V'(u)h = \lim_{n \to +\infty} V'(u_n)h$.

Proof. Since, by $(f2)$, $p+1<2_N^*$, we have $H_r^1(\mathbb{R}^N)$ is compactly embedded in $L^{p+1}(\mathbb{R}^N)$ and so $u_n \to u$ and $v_n \to v$ strongly in $L^{p+1}(\mathbb{R}^N)$. Then, since by (1.3) we have that for all $\varepsilon > 0$

$$\begin{aligned}|F(u_n)-F(u)| &= |\int_0^1 f(u+s(u_n-u))(u_n-u)\,ds| \\ &\le \varepsilon|u_n-u|(|u|+|u_n-u|)+2^{p-1}A_\varepsilon|u_n-u|(|u|^p+|u_n-u|^p),\end{aligned}$$

we deduce that as $n\to+\infty$

$$\begin{aligned}&\int_{\mathbb{R}^N}|F(u_n)-F(u)|\,dx \le \varepsilon\|u_n-u\|_2(\|u\|_2+\|u_n-u\|_2) \\ &\quad + 2^{p-1}A_\varepsilon\|u_n-u\|_{p+1}(\|u\|_{p+1}^p+\|u_n-u\|_{p+1}^p) = \varepsilon\|u_n-u\|_2(\|u\|_2+\|u_n-u\|_2)+o(1).\end{aligned}$$

Since ε is arbitrary and $\|u_n-u\|$ bounded, we deduce that $\int_{\mathbb{R}^N}F(u_n)\,dx \to \int_{\mathbb{R}^N}F(u)dx$. We show now that $\int_{\mathbb{R}^N}f(u_n)v_n\,dx \to \int_{\mathbb{R}^N}f(u)vdx$. First we write

$$\int_{\mathbb{R}^N}f(u_n)v_n\,dx = \int_{\mathbb{R}^N}f(u_n)(v_n-v)dx+\int_{\mathbb{R}^N}f(u_n)vdx. \tag{2.5}$$

We note that by (1.3), since $\|v_n-v\|_{p+1}\to 0$, as $n\to+\infty$ we have

$$\begin{aligned}|\int_{\mathbb{R}^N}f(u_n)(v_n-v)dx| &\le \int_{\mathbb{R}^N}\varepsilon|u_n||v_n-v|+A_\varepsilon|u_n|^p|v_n-v|\,dx \\ &\le \varepsilon\|u_n\|_2\|v_n-v\|_2+A_\varepsilon\|u_n\|_{p+1}^p\|v_n-v\|_{p+1} = \varepsilon\|u_n\|_2\|v_n-v\|_2+o(1)\end{aligned}$$

and so, since ε is arbitrary, we deduce $\int_{\mathbb{R}^N}f(u_n)(v_n-v)dx\to 0$. Then, by (2.5), our claim follows if we show that $\int_{\mathbb{R}^N}f(u_n)vdx \to \int_{\mathbb{R}^N}f(u)vdx$. For that we fix $\varepsilon>0$ and choose $R>0$ such that $\int_{|x|>R}|v|^2\,dx+\int_{|x|>R}|v|^{p+1}\,dx<\varepsilon$. By (1.3), with $\varepsilon=1$, we have

$$\begin{aligned}|\int_{|x|>R}f(u_n)v-f(u)v\,dx| &\le \int_{|x|>R}(|u_n|+|u|)|v|+A_1(|u|^p+|u_n|^p)|v|\,dx \\ &\le \varepsilon^{1/2}(\|u_n\|_2+\|u\|_2)+A_1\varepsilon^{1/(p+1)}(\|u_n\|_{p+1}^p+\|u\|_{p+1}^p)\end{aligned}$$

and, since ε is arbitrary, we deduce $\int_{|x|>R}f(u_n)v\,dx \to \int_{|x|>R}f(u)v\,dx$. On the other hand we have $u_n\to u$ in $L^2(B_R(0))$ and in $L^{p+1}(B_R(0))$. Then for any subsequence (u_{n_k}) there exists a subsubsequence $(u_{n_{k_j}})$ and a function $\psi\in L^2(B_R(0))\cap L^{p+1}(B_R(0))$ such that $u_{n_{k_j}}(x)\to u(x)$ a.e. in $B_R(0)$ and $|u_{n_{k_j}}(x)|\le\psi(x)$ a.e. in $B_R(0)$, for any $j\in\mathbb{N}$. Using again (1.3) we obtain $|f(u_{n_{k_j}})-f(u)||v|\le(|\psi|+|u|)|v|+A_1(|\psi|^p+|u|^p)|v|$ on $B_R(0)$ for any $j\in\mathbb{N}$. Then, by the dominated convergence theorem, we get $\int_{|x|<R}f(u_{n_{k_j}})v\,dx\to\int_{|x|<R}f(u)v\,dx$. Since the subsequence (u_{n_k}) is arbitrary we conclude that $\int_{|x|<R}f(u_n)v\,dx\to\int_{|x|<R}f(u)v\,dx$ and the Lemma follows. □

For our study it is important to understand the structure of the sublevel sets $\mathcal{V}^b=\{u\in H_r^1(\mathbb{R}^N)\,/\,V(u)\le b\}$. By definition of the Mountain pass level the set $\mathcal{V}^b$ is not path connected for any $b\in[0,c)$. Given $b\in[0,c)$, recalling Lemma 2.3, we denote

$\mathcal{V}_-^b=\{tu\,|\,u\in H_r^1(\mathbb{R}^N)\setminus\{0\},\,t\in[0,\alpha_{u,b}]\}$ and $\mathcal{V}_+^b=\{tu\,|\,u\in H_r^1(\mathbb{R}^N)\setminus\{0\},\,t\in[\omega_{u,b},+\infty)\}$.

Clearly

$$\mathcal{V}^b=\mathcal{V}_-^b\cup\mathcal{V}_+^b.$$

Remark 2.3 The set $\mathcal{V}_-^b$ is clearly path connected (starshaped indeed, with respect to the origin). The same holds true also for $\mathcal{V}_+^b$. Indeed, given $u_1, u_2 \in \mathcal{V}_+^b$ such that $b \geq b_1 = V(u_1) \geq b_2 = V(u_2)$ we can connect them considering the path $\gamma(s) = \omega_{b_1,(1-s)u_1+su_2}((1-s)u_1+su_2)$ for $s \in [0,1]$ and $\gamma(s) = s\omega_{b_1.u_2}$ for $s \in [1, 1/\omega_{b_1,u_2}]$. The function γ is continuous since the mapping $u \in H_r^1(\mathbb{R}^N) \to \omega_{u,b} \in \mathbb{R}$ is continuous for any $b < c$.

Remark 2.4 By definition of mountain pass level and Remark 2.3, if $\gamma \in C([0,1], H_r^1(\mathbb{R}^N))$ is such that $\gamma(0) \in \mathcal{V}_-^b$ and $\gamma(1) \in \mathcal{V}_+^b$ then $\max_{s\in[0,1]} V(\gamma(s)) \geq c$. Secondly note that by Lemma 2.3

$$\mathcal{V}_-^b = \{u \in H_r^1(\mathbb{R}^N) \,/\, \alpha_{u,b} \geq 1\} \cup \{0\} \text{ and } \mathcal{V}_+^b = \{u \in H_r^1(\mathbb{R}^N) \,/\, \omega_{u,b} \leq 1\} \text{ for all } b \in [0,c).$$

Moreover if $b \in (0,c)$ then

$$u \in \mathcal{V}_-^b \setminus \{0\} \text{ if and only if } V(u) \leq b \text{ and } V'(u)u > 0. \tag{2.6}$$

Indeed, if $u \in \mathcal{V}_-^b \setminus \{0\}$ then $1 \leq \alpha_{u,b} < t_u$ and so, by Lemma 2.3, $V'(u)u > 0$. Viceversa if $V(u) \leq b$ and $V'(u)u > 0$ then $u \neq 0$ and $1 \leq \alpha_{u,b}$, from which $V(u) \leq b$. Analogously if $b \in [0,c)$ then

$$u \in \mathcal{V}_+^b \text{ if and only if } V(u) \leq b \text{ and } V'(u)u < 0. \tag{2.7}$$

Lemma 2.5 *If $b \in [0,c)$ then $\mathcal{V}_-^b$ and $\mathcal{V}_+^b$ are weakly closed in $H_r^1(\mathbb{R}^N)$.*

Proof. Let $(u_n) \subset \mathcal{V}_+^b$ be such that $u_n \to u_0$ weakly in $H_r^1(\mathbb{R}^N)$. By Remark 2.4 we have $V(u_n) \leq b$ and $V'(u_n)u_n < 0$. Since $V'(u_n)u_n < 0$, by Lemma 2.2 we deduce $\|u_n\| \geq \rho$ for any $n \in \mathbb{N}$. Moreover since $V(u_n) \leq b$, by Lemma 2.4 we obtain $V(u_0) \leq b$. By Lemma 2.4 we know also that $\int_{\mathbb{R}^n} f(u_n)u_n\,dx \to \int_{\mathbb{R}^n} f(u_0)u_0\,dx$ and, since $V'(u_n)u_n < 0$, $V'(u_0)u_0 \leq 0$. By (2.7), to prove that $u_0 \in \mathcal{V}_+^b$ we have to show that $V'(u_0)u_0 < 0$. For that, assume by contradiction that $V'(u_0)u_0 = 0$ and note that, being $V(u_0) \leq b < c$, by Remark 2.1 we have $u_0 = 0$. Then $\int_{\mathbb{R}^n} f(u_n)u_n\,dx \to 0$ and so $0 > V'(u_n)u_n > \rho^2 + o(1)$ as $n \to +\infty$, a contradiction which shows that $\mathcal{V}_+^b$ is weakly closed.
Let now $(u_n) \subset \mathcal{V}_-^b$ be such that $u_n \to u_0$ weakly in $H_r^1(\mathbb{R}^N)$. Again using Remark 2.4 we have $V(u_n) \leq b$ and $V'(u_n)u_n \geq 0$. Hence, by Lemma 2.4, we deduce that $V(u_0) \leq b$. To show that $u_0 \in \mathcal{V}_-^b$ it suffices to show that $V'(u_0)u_0 \geq 0$. Assume by contradiction that $V'(u_0)u_0 < 0$. Then, by (2.7), we have $u_0 \in \mathcal{V}_+^b$. Consider the path $\gamma_n(s) = u_0 + s(u_n - u_0)$, $s \in [0,1]$. Since $\gamma_n(0) = u_0 \in \mathcal{V}_+^b$ and $\gamma_n(1) = u_n \in \mathcal{V}_-^b$, by Remark 2.4, for any $n \in \mathbb{N}$ we find $s_n \in (0,1)$ such that $V(\gamma_n(s_n)) \geq c$. We note also that $\|\gamma_n(s)\|_2 \leq \|u_0\|_2 + \|u_n - u_0\|_2 \leq C_1 < +\infty$ and $\|\gamma_n(s)\|_{p+1} \leq \|u_0\|_{p+1} + \|u_n - u_0\|_{p+1} \leq C_2 < +\infty$ for any $n \in \mathbb{N}$ and $s \in [0,1]$. Then, choosing $\varepsilon = \frac{c-b}{2C_1^2}$, by (1.3) we get

$$\begin{aligned} \Big|\int_{\mathbb{R}^N} f(\gamma_n(s))(u_n - u_0)dx\Big| &\leq \varepsilon\|\gamma_n(s)\|_2\|u_n - u_0\|_2 + A_\varepsilon\|\gamma_n(s)\|_{p+1}^p\|u_n - u_0\|_{p+1} \\ &= \tfrac{c-b}{2} + A_\varepsilon C_2^p\|u_n - u_0\|_{p+1} \quad \text{for any } s \in [0,1]. \end{aligned}$$

Hence we derive that for any $s \in [0,1]$ and $n \in \mathbb{N}$ there results

$$\begin{aligned} \tfrac{d}{ds}V(\gamma_n(s)) &= V'(\gamma_n(s))(u_n - u_0) \\ &\geq s\|u_n - u_0\|^2 + \langle u_0, u_n - u_0\rangle - \tfrac{c-b}{2} - A_\varepsilon C_2^p\|u_n - u_0\|_{p+1}. \end{aligned}$$

Integrating on $[s_n, 1]$ we get

$$b - c \geq V(u_n) - V(\gamma_n(s_n)) \geq \tfrac{b-c}{2} + (1 - s_n)(\langle u_0, u_n - u_0\rangle - A_\varepsilon C_2^p\|u_n - u_0\|_{p+1}).$$

Since $\langle u_0, u_n - u_0\rangle - A_\varepsilon C_2^p\|u_n - u_0\|_{p+1} \to 0$ we obtain the contradiction $0 > b - c \geq \frac{b-c}{2}$. $\square$

Remark 2.5 Note that, by (2.2), if $b \in [0,c)$ and $u \in \mathcal{V}_-^b$, since $V'(u)u \geq 0$, then

$$\|u\|^2 \leq \tfrac{2\mu}{\mu-2} V(u) \leq \tfrac{2\mu}{\mu-2} b.$$

In particular we obtain that $\mathcal{V}_-^b$ is bounded in $H_r^1(\mathbb{R}^N)$. Then, by Lemma 2.5, $\mathcal{V}_-^b$ is weakly compact in $H_r^1(\mathbb{R}^N)$ and if $(u_n) \subset \mathcal{V}_-^b$ is such that $u_n \to u_0$ with respect to the $L^2(\mathbb{R}^N)$ metric then $u_0 \in \mathcal{V}_-^b$.

Lemma 2.6 *If $b \in [0,c)$ we have $\nu^+(b) := \inf_{u \in \mathcal{V}_+^b} \frac{-V'(u)u}{\max\{1, \|u\|_2^2\}} > 0$.*

Proof. First note that, by (2.2), if $u \in \mathcal{V}_+^b$ is such that $\|u\|_2^2 \geq \frac{4b\mu}{\mu-2}$ or $V(u) \leq 0$ then $\frac{-V'(u)u}{\|u\|_2^2} \geq \frac{\mu-2}{2}\frac{\|u\|^2}{\|u\|_2^2} - \mu\frac{V(u)}{\|u\|_2^2} \geq \frac{\mu-2}{4}$. Assume now by contradiction that there exists $(u_n) \subset \mathcal{V}_+^b$ such that $0 < V(u_n) \leq b$, $\|u_n\|_2^2 \leq \frac{4b\mu}{\mu-2}$ and $\frac{-V'(u_n)u_n}{\max\{1,\|u_n\|_2^2\}} \to 0$. Then $V'(u_n)u_n \to 0$. Since $u_n \in \mathcal{V}_+^b$, by Remark 2.4 we have $t_{u_n} < 1$. By (2.2) we have $\|u_n\|^2 \leq \frac{2\mu}{\mu-2}b + o(1)$ and since, by Remark 2.1, $\|t_{u_n}u_n\| \geq \rho$, we deduce that $t_{u_n} \geq \frac{\mu-2}{4\mu b}\rho > 0$ whenever n is large. By Lemma 2.3 we have $|V'(su_n)su_n| \leq |V'(u_n)u_n|$ for any $s \in (t_{u_n}, 1)$, and we conclude $c - b \leq \int_1^{t_{u_n}} \frac{d}{ds}V(su_n)\,ds = \int_1^{t_{u_n}} \frac{1}{s}V'(su_n)su_n\,ds \leq -\log(t_{u_n})|V'(u_n)u_n| \to 0$ as $n \to +\infty$, a contradiction which proves the Lemma. $\square$

Lemma 2.7 *If $b \in (0,c)$ then $\nu^-(b) := \inf_{u \in \mathcal{V}_-^{(b+c)/2} \setminus \mathcal{V}_-^b} V'(u)u > 0$.*

Proof. By contradiction, let $(u_n) \subset \mathcal{V}_-^{(b+c)/2} \setminus \mathcal{V}_-^b$ be such that $V'(u_n)u_n \to 0$. Then, by Remark 2.5, there exists $u_0 \in \mathcal{V}_-^{(b+c)/2}$ such that, up to a subsequence, $u_n \to u_0$ weakly in $H_r^1(\mathbb{R}^N)$. By Lemma 2.4, $V'(u_0)u_0 \leq \liminf V'(u_n)u_n = 0$. Since $u_0 \in \mathcal{V}_-^{(b+c)/2}$ that implies $u_0 = 0$ and then, again by Lemma 2.4, $\int_{\mathbb{R}^n} f(u_n)u_n\,dx \to 0$. Hence $V'(u_n)u_n = \|u_n\|^2 + o(1) \to 0$ and so $u_n \to 0$ in $H^1(\mathbb{R}^n)$ that gives the contradiction $0 < b \leq V(u_n) \to 0$. $\square$

Finally, we display some properties depending on the assumption $p < 1 + \frac{4}{N}$.

First, as a particular case of the Gagliardo Nirenberg interpolation inequality (see [24]), we have that there exists a constant $\kappa = \kappa(N,p) > 0$ such that for any $u \in H_r^1(\mathbb{R}^N)$, there results

$$\|u\|_{p+1} \leq \kappa \|u\|_2^\theta \|\nabla u\|_2^{1-\theta}, \quad \text{where } 1 - \theta = \tfrac{N}{2}\tfrac{p-1}{p+1}. \tag{2.8}$$

Moreover, note that, by (1.4), we have $F(t) \leq \frac{1}{4}|t|^2 + \frac{A_{1/2}}{p+1}|t|^{p+1}$ for every $t \in \mathbb{R}$. Therefore, if $u \in H_r^1(\mathbb{R}^N) \setminus \{0\}$, by (2.8) there results

$$V(u) \geq \tfrac{1}{2}\|\nabla u\|_2^2 \left(1 - \tfrac{2\kappa_{GN} A_{1/2}}{p+1} \tfrac{\|u\|_2^{(p+1)\theta}}{\|\nabla u\|_2^{2-(p+1)(1-\theta)}}\right) + \tfrac{1}{4}\|u\|_2^2. \tag{2.9}$$

where, since $p < 1 + \frac{4}{N}$, by $(f2)$, we have

$$(p+1)(1-\theta) = \tfrac{N}{2}(p-1) < 2. \tag{2.10}$$

By (2.9) and (2.10) it follows directly

Lemma 2.8 *If $(u_n) \subset H_r^1(\mathbb{R}^N)$, $\sup\limits_{n \in \mathbb{N}} \|u_n\|_2 < +\infty$ and $\|\nabla u_n\|_2 \to +\infty$ then $V(u_n) \to +\infty$.*

In particular $\mathcal{V}^b_+$ enjoys the following property.

Lemma 2.9 *If $b \in [0, c)$, for any $M_1 > 0$ there exists $M_2 > 0$ such that if $u \subset \mathcal{V}^b_+$ and $\|u\|_2 \leq M_1$ then $\|\nabla u\|_2 \leq M_2$.*

Remark 2.6 Note that by Lemma 2.9 and Lemma 2.5 we derive that if $(u_n) \subset \mathcal{V}^b_+$ is such that $u_n \to u_0$ with respect to the $L^2(\mathbb{R}^N)$ metric then $u_0 \in \mathcal{V}^b_+$.

Another consequence is the following one

Lemma 2.10 *For any b_1, $b_2 \in [0, c)$ there result $\delta(b_1, b_2) := \mathrm{dist}(\mathcal{V}^{b_1}_-, \mathcal{V}^{b_2}_+) > 0$.*

Proof. Clearly $\delta(b_1, b_2) < +\infty$. Let $(u_{n,1}) \subset \mathcal{V}^{b_1}_-$ and $(u_{n,2}) \subset \mathcal{V}^{b_2}_+$ be such that $\|u_{n,1} - u_{n,2}\|_2 \to \delta(b_1, b_2)$. By Remark 2.5 we know that $\|u_{n,1}\| \leq \frac{2\mu}{\mu-2} b_1$ and hence we obtain $\|u_{n,2}\|_2 \leq \frac{2\mu}{\mu-2} b_1 + \delta(b_1, b_2) + o(1)$. Then $(u_{n,2})$ is bounded in $L^2(\mathbb{R})$. By Lemma 2.8, since $V(u_{n,2}) \leq b_2$, we obtain that $\sup_{n\in\mathbb{N}} \|\nabla u_{n,2}\|_2 < +\infty$ and so that $(u_{n,2})$ is bounded also in $H^1_r(\mathbb{R}^N)$. Then there exists two subsequences $(u_{n_j,1}) \subset (u_{n,1})$, $(u_{n_j,2}) \subset (u_{n,2})$ which weakly converge respectively to $u_1 \in H^1_r(\mathbb{R}^N)$ and $u_2 \in H^1_r(\mathbb{R}^N)$. By Lemma 2.5 we have $u_1 \in \mathcal{V}^{b_1}_-$ and $u_2 \in \mathcal{V}^{b_2}_+$ and by the weak semicontinuity of the L^2 norm we deduce $\delta(b_1, b_2) \leq \|u_1 - u_2\|_2 \leq \lim_{j\to+\infty} \|u_{n_j,1} - u_{n_j,2}\|_2 = \delta(b_1, b_2)$. Since $u_1 \neq u_2$ we have $\delta(b_1, b_2) = \|u_1 - u_2\|_2 > 0$ and the Lemma follows. $\square$

As a further consequence of the assumption $p < 1 + 4/N$, we give a result concerning the behaviour of V along sequences in $H^1_r(\mathbb{R}^N)$ which converge to a point $u_0 \in H^1_r(\mathbb{R}^N)$ with respect to the $L^2(\mathbb{R}^N)$ metric.

Lemma 2.11 *Let $u_n, u_0 \in H^1_r(\mathbb{R}^N)$ be such that $\|u_n - u_0\|_2 \to 0$ as $n \to +\infty$ and $\liminf_{n\to\infty} \|\nabla(u_n - u_0)\|_2 > 0$. Then there exists $\bar{n} \in \mathbb{N}$ such that*

$$V(u_n) - V(u_0 + s(u_n - u_0)) \geq \tfrac{1}{4}(1 - s)\|\nabla(u_n - u_0)\|^2_2, \qquad \forall s \in [0, 1],\ n \geq \bar{n}.$$

Proof. Setting $w_n = u_n - u_0$, by (1.3), since $w_n \to 0$ in $L^2(\mathbb{R}^N)$ we recover that there exists $C > 0$ such that, for any $s \in [0, 1]$,

$$\begin{aligned}
&|\int_{\mathbb{R}^N} F(u_0 + w_n) - F(u_0 + sw_n))\, dx| = |\int_{\mathbb{R}^N} \int_s^1 f(u_0 + \sigma w_n) w_n d\sigma\, dx| \\
&\leq \int_s^1 \|u_0\|_2 \|w_n\|_2 + \sigma \|w_n\|^2_2 + A_1 2^{p-1}(\|u_0\|^p_{p+1} \|w_n\|_{p+1} + \sigma^p \|w_n\|^{p+1}_{p+1})\, d\sigma \\
&\leq C(1 - s)(o(1) + \|w_n\|_{p+1} + \|w_n\|^{p+1}_{p+1}) \qquad \text{as } n \to +\infty. \qquad (2.11)
\end{aligned}$$

We now note that, since $\liminf_{n\to+\infty} \|\nabla w_n\|^2_2 > 0$, we have

$$\lim_{n\to+\infty} \frac{\langle \nabla u_0, \nabla w_n \rangle_2}{\|\nabla w_n\|^2_2} = 0. \qquad (2.12)$$

Indeed, (2.12) is true along subsequences (w_{n_j}) such that $\|\nabla w_{n_j}\|_2 \to +\infty$. If $(w_{n_j}) \subset \{w_n\}$ is bounded in $H^1_r(\mathbb{R}^N)$ then, necessarily, $w_{n_j} \to 0$ weakly in $H^1_r(\mathbb{R}^N)$ and again (2.12) follows.

Secondly we note that

$$\lim_{n\to+\infty} \frac{\|w_n\|_{p+1} + \|w_n\|^{p+1}_{p+1}}{\|\nabla w_n\|^2_2} = 0. \qquad (2.13)$$

Indeed, we have either $\|\nabla w_n\|_2$ is bounded or $\limsup_{n\to+\infty} \|\nabla w_n\|_2 = +\infty$. If $\|\nabla w_n\|_2$ is bounded then (w_n) weakly converges to 0 in $H^1_r(\mathbb{R}^N)$ and so strongly in $L^{p+1}(\mathbb{R}^N)$ giving

(2.13). If $\|\nabla w_n\|_2 \to +\infty$ along a subsequence, then, since $\|w_n\|_2 \to 0$, (2.13) follows by (2.8) and (2.10).

Finally, by (2.11), we derive that for any $s \in [0,1]$

$$V(u_0+w_n) - V(u_0+sw_n) = \tfrac{\|\nabla w_n\|^2}{2}(1-s^2) + (1-s)\langle \nabla u_0, \nabla w_n\rangle_2 + (1-s)o(1) \\ - \int_{\mathbb{R}^N} F(u_0+w_n) - F(u_0+sw_n)\, dx$$

$$\geq \|\nabla w_n\|_2^2(1-s)\left(\tfrac{1+s}{2} + \tfrac{\langle \nabla u_0, \nabla w_n\rangle_2}{\|\nabla w_n\|_2^2} - C\tfrac{\|w_n\|_{p+1}+\|w_n\|_{p+1}^{p+1}+o(1)}{\|\nabla w_n\|_2^2}\right) \geq \|\nabla w_n\|_2^2(1-s)\left(\frac{1}{2}+o(1)\right)$$

and the Lemma follows by (2.12) and (2.13). □

Remark 2.7 By Lemma 2.11 we have in particular that if $u_n, u_0 \in H^1_r(\mathbb{R}^N)$, $s_n \in [0,1]$ are such that $u_n \to u_0$ in $L^2(\mathbb{R}^N)$ as $n \to +\infty$ and $V(u_n) - V(u_0 + s_n(u_n - u_0)) \to 0$ as $n \to +\infty$, then $(1-s_n)\|u_n - u_0\|^2 \to 0$ as $n \to +\infty$. In particular, if $V(u_n) \to V(u_0)$ as $n \to +\infty$, then $u_n \to u_0$ in $H^1(\mathbb{R}^N$ as $n \to +\infty$.

3 Solutions on $\mathbb{R}^{N+1}$.

In the sequel we denote $(x,y) \in \mathbb{R}^{N+1}$ where $x = (x_1,\ldots,x_n) \in \mathbb{R}^N$ and $y \in \mathbb{R}$, the gradient with respect to the $x \in \mathbb{R}^N$ will be denoted by ∇_x. For $(y_1,y_2) \subset \mathbb{R}$ we set $S_{(y_1,y_2)} := \mathbb{R}^N \times (y_1,y_2)$ and, more simply, $S_L := S_{[-L,L]}$ for $L > 0$. We denote by $\mathcal{X}$ the set of monotone decreasing radially symmetric functions in $H^1(\mathbb{R}^N)$:

$$\mathcal{X} = \{u \in H^1_r(\mathbb{R}^N) \,|\, u(x_1) \geq u(x_2) \text{ for any } x_1,\, x_2 \in \mathbb{R}^N \text{ such that } |x_1| \leq |x_2|\}.$$

Note that $\mathcal{X}$ is a positive cone in $H^1_r(\mathbb{R}^N)$ (and so convex) and it is sequentially closed in $H^1(\mathbb{R}^N)$ with respect to the weak topology. In the following, with abuse of notation, given $b \in [0,c)$ we will indicate $\mathcal{V}^b_\pm \equiv \mathcal{V}^b_\pm \cap \mathcal{X}$.

We consider the set

$$\mathcal{H} = \{v \in \cap_{L>0} H^1(S_L) \,/\, v(\cdot,y) \in \mathcal{X} \text{ for a.e. } y \in \mathbb{R}\}.$$

Note that, by Fubini Theorem, we have that if $v \in \mathcal{H}$ then $v(x,\cdot) \in H^1_{loc}(\mathbb{R})$ for a.e. $x \in \mathbb{R}^N$. Therefore, if $(y_1,y_2) \subset \mathbb{R}$ then $v(x,y_2) - v(x,y_1) = \int_{y_1}^{y_2} \partial_y v(x,y)\, dy$ holds for a.e. $x \in \mathbb{R}^N$ and so

$$\int_{\mathbb{R}^N} |v(x,y_2) - v(x,y_1)|^2\, dx = \int_{\mathbb{R}^N} |\int_{y_1}^{y_2} \partial_y u(x,y)\, dy\,|^2 dx \leq |y_2 - y_1| \int_{\mathbb{R}^N}\int_{y_1}^{y_2} |\partial_y v(x,y)|^2\, dy dx$$

According to that, if $v \in \mathcal{H}$, the function $y \in \mathbb{R} \mapsto u(\cdot,y) \in L^2(\mathbb{R}^N)$, defines a continuous trajectory verifying

$$\|v(\cdot,y_2) - v(\cdot,y_1)\|_2^2 \leq \|\partial_y v\|^2_{L^2(S(y_1,y_2))}|y_2 - y_1|, \quad \forall\,(y_1,y_2) \subset \mathbb{R}. \tag{3.1}$$

In the sequel we will consider the functional V as extended on $L^2(\mathbb{R}^N)$ in the following way

$$V(u) = \begin{cases} V(u), & \text{if } u \in H^1(\mathbb{R}^n), \\ +\infty, & \text{if } u \in L^2(\mathbb{R}^N) \setminus H^1(\mathbb{R}^n). \end{cases}$$

Lemma 3.1 *If $v \in \mathcal{H}$ then the function $y \in \mathbb{R} \to V(v(\cdot, y)) \in \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous.*

Proof. Let $v \in \mathcal{H}$ and $y_n \to y_0$ and let $(y_{n_j}) \subset (y_n)$ be such that $\liminf_{n\to+\infty} V(v(\cdot, y_n)) = \lim_{j\to+\infty} V(v(\cdot, y_{n_j}))$. By (3.1) we have $v(\cdot, y_{n_j}) \to v(\cdot, y_0)$ in $L^2(\mathbb{R}^N)$ as $j \to +\infty$. We consider the two following alternative case:

$$\text{(a)}\ \sup_{j\in\mathbb{N}} \|v(\cdot, y_{n_j})\| < +\infty \quad \text{or} \quad \text{(b)}\ \limsup_{j\to+\infty} \|v(\cdot, y_{n_j})\| = +\infty$$

In the case (a), since $(v(\cdot, y_{n_j}))$ is bounded in $\mathcal{X}$ and $v(\cdot, y_{n_j}) \to v(\cdot, y_0)$ in $L^2(\mathbb{R}^N)$, we deduce that $v(\cdot, y_{n_j}) \to v(\cdot, y_0)$ weakly in $\mathcal{X}$. Then by Lemma 2.4 we derive $\lim_{j\to+\infty} V(v(\cdot, y_{n_j})) \geq V(v(\cdot, y_0))$. In the case (b) we have $\limsup_{j\to+\infty} \|\nabla v(\cdot, y_{n_j})\|_2 = +\infty$ since $\|v(\cdot, y_{n_j})\|_2$ is bounded. Then, by Lemma 2.8, we get $\lim_{j\to+\infty} V(v(\cdot, y_{n_j})) = \limsup_{j\to+\infty} V(v(\cdot, y_{n_j})) = +\infty$, showing that also in the case (b) there results $\lim_{j\to+\infty} V(v(\cdot, y_{n_j})) \geq V(v(\cdot, y_0))$. $\square$

Lemma 3.2 *If $v \in \mathcal{H}$ is a solution of (E) on $S_{(y_1,y_2)}$ then the energy function $E_v(y) = \frac{1}{2}\|\partial_y v(\cdot, y)\|_2^2 - V(v(\cdot, y))$ is constant on (y_1, y_2).*

Proof. Since $v \in \mathcal{H}$ we have that $v \in H^1(S_L)$ for any $L > 0$. Then, since v solve (E) on $S_{(y_1,y_2)}$ by regularity we have $v \in H^2(S_{(\zeta_1,\zeta_2)}) \cap C^2(S_{(y_1,y_2)})$ for any $[\zeta_1, \zeta_2] \subset (y_1, y_2)$. Hence $v(\cdot, y) \in H^2(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$ for all $y \in (y_1, y_2)$ and so $\int_{|x|=R} |v(x, y)| + |\nabla_x v(x, y)|\, d\sigma \to 0$ as $R \to +\infty$ for all $y \in (y_1, y_2)$. Denoting $\mathrm{div}_x w = \sum_{i=1}^n \partial_{x_i} w$, we derive

$$\int_{\mathbb{R}^N} \mathrm{div}_x[\partial_y v \nabla_x v]\, dx = \lim_{R\to+\infty} \int_{|x|\leq R} \mathrm{div}_x[\partial_y v \nabla_x v]\, dx = \lim_{R\to+\infty} \int_{|x|=R} \partial_y v \nabla_x v \cdot \tfrac{x}{|x|}\, d\sigma = 0.$$

Therefore, multiplying (E) by $\partial_y v$ and integrating over $\mathbb{R}^N$ with respect to x, we obtain

$$\begin{aligned}
0 &= \int_{\mathbb{R}^N} -\partial_y^2 v \partial_y v - \Delta_x v \partial_y v + v \partial_y v - f(v) \partial_y v\, dx \\
&= \int_{\mathbb{R}^N} -\tfrac{1}{2}\partial_y |\partial_y v|^2 - \mathrm{div}_x[\partial_y v \nabla_x v] + \tfrac{1}{2}\partial_y |\nabla_x v|^2 + \partial_y(\tfrac{1}{2}|v|^2 - F(v))\, dx \\
&= \partial_y \left[\int_{\mathbb{R}^N} -\tfrac{1}{2}|\partial_y v|^2 + \tfrac{1}{2}|\nabla_x v|^2 + \tfrac{1}{2}|v|^2 - F(v)\, dx \right] \\
&= \partial_y \left[-\tfrac{1}{2}\|\partial_y v(\cdot, y)\|_2^2 + V(v(\cdot, y)) \right] = -\partial_y E_v(y)
\end{aligned}$$

and the Lemma follows. $\square$

3.1 The variational setting

Fixed $b \in [0, c)$ we consider the space

$$\mathcal{X}_b = \{v \in \mathcal{H} \,/\, \liminf_{y\to\pm\infty} \mathrm{dist}(v(\cdot, y), \mathcal{V}^b_\pm) = 0 \text{ and } \inf_{y\in\mathbb{R}} V(v(\cdot, y)) \geq b\}$$

on which we look for minima of the functional

$$\varphi(v) = \int_{\mathbb{R}} \tfrac{1}{2}\|\partial_y v(\cdot, y)\|_2^2 + (V(v(\cdot, y)) - b)\, dy.$$

Remark 3.1 The problem of finding a minimum of φ on $\mathcal{X}_b$ is well posed. In fact, if $v \in \mathcal{X}_b$ then $V(v(\cdot, y)) \geq b$ for every $y \in \mathbb{R}$ and so the functional φ is well defined and non negative on $\mathcal{X}_b$. Moreover $\mathcal{X}_b \neq \emptyset$ and $m_b = \inf_{v \in \mathcal{X}_b} \varphi(v) < +\infty$. Indeed, for any $u \in \mathcal{X}$, recalling Lemma 2.3 and considered the function

$$v(x,y) = \begin{cases} \omega_{b,u} u(x) & x \in \mathbb{R}^N,\ y \geq \omega_{b,u}, \\ y u(x) & x \in \mathbb{R}^N,\ \alpha_{u,b} < y < \omega_{b,u}, \\ \alpha_{u,b} u(x) & x \in \mathbb{R}^N,\ y < \alpha_{u,b}, \end{cases}$$

we have that $v \in \mathcal{X}_b$ and $\varphi(v) = \int_{\alpha_{u,b}}^{\omega_{u,b}} \frac{1}{2}\|u\|_2^2 + V(yu) - b\, dy \leq (\frac{1}{2}\|u\|_2^2 + V(t_u u) - b)(\omega_{u,b} - \alpha_{u,b}) < +\infty$.

Remark 3.2 More generally, given an interval $I \subset \mathbb{R}$ we consider the functional

$$\varphi_I(v) = \int_I \tfrac{1}{2}\|\partial_y v(\cdot, y)\|_2^2 + V(v(\cdot, y)) - b\, dy$$

which is well defined for any $v \in \mathcal{H}$ such that $V(v(\cdot, y)) \geq b$ for a.e. $y \in I$ or for every $v \in \mathcal{H}$ if I is bounded.

We will make use of the following semicontinuity property

Lemma 3.3 *Let $v \in \mathcal{H}$ be such that $V(v(\cdot, y)) \geq b$ for a.e. $y \in I \subset \mathbb{R}$. If $(v_n) \subset \mathcal{X}_b$ is such that $v_n \to v$ weakly in $H^1(S_L)$ for any $L > 0$, then $\varphi_I(v) \leq \liminf_{n\to\infty} \varphi_I(v_n)$.*

Proof. Let $L_1 < L_2 \in \mathbb{R}$ be such that $(L_1, L_2) \subset I$. The sequence (v_n) is weakly convergent to v in $H^1(S_{(L_1,L_2)})$ and constituted by radially symmetric functions in the x variable. By Lemma III.2 in [19] we derive that $v_n \to v$ strongly in $L^{p+1}(S_{(L_1,L_2)})$. Then, since by (1.3) we have $|F(v_n) - F(v)| \leq \varepsilon |v_n - v|(|v| + |v_n - v|) + 2^{p-1} A_\varepsilon |v_n - v|(|v|^p + |v_n - v|^p)$, we deduce that, as $n \to +\infty$,

$$\begin{aligned} \int_{S_{(L_1,L_2)}} |F(v_n) - F(v)|\, dxdy &\leq \varepsilon \|v_n - v\|_{L^2(S_{(L_1,L_2)})}(\|v\|_{L^2(S_{(L_1,L_2)})} + \|v_n - v\|_{L^2(S_{(L_1,L_2)})}) \\ &\quad + 2^{p-1} A_\varepsilon \|v_n - v\|_{L^{p+1}(S_{(L_1,L_2)})}(\|v\|^p_{L^{p+1}(S_{(L_1,L_2)})} + \|v_n - v\|^p_{L^{p+1}(S_{(L_1,L_2)})}) \\ &= \varepsilon \|v_n - v\|_{L^2(S_{(L_1,L_2)})}(\|v\|_{L^2(S_{(L_1,L_2)})} + \|v_n - v\|_{L^2(S_{(L_1,L_2)})}) + o(1) \end{aligned}$$

Since ε is arbitrary and $(v_n - v)$ is bounded in $L^2(S_{(L_1,L_2)})$, we deduce $\int_{S_{(L_1,L_2)}} F(v_n)\, dxdy \to \int_{S_{(L_1,L_2)}} F(v) dxdy$. Then, by the weak semicontinuity of the norm, we obtain

$$\begin{aligned} \liminf_{n\to+\infty} \varphi_I(v_n) &\geq \liminf_{n\to+\infty} \varphi_{(L_1,L_2)}(v_n) \\ &= \liminf_{n\to+\infty} \tfrac{1}{2}\|v_n\|^2_{H^1(S_{(L_1,L_2)})} - \int_{S_{(L_1,L_2)}} F(v_n) dxdy - b(L_2 - L_1) \\ &\geq \tfrac{1}{2}\|v\|^2_{H^1(S_{(L_1,L_2)})} - \int_{S_{(L_1,L_2)}} F(v) dxdy - b(L_2 - L_1) = \varphi_{(L_1,L_2)}(v) \end{aligned}$$

and the Lemma follows by the arbitrariness of L_1 and L_2. $\square$

Remark 3.3 In the sequel we will study coerciveness properties of φ. One of the key tools is the following simple estimate. Given $v \in \mathcal{H}$, and $(y_1, y_2) \subset \mathbb{R}$ we have

$$\begin{aligned}
\varphi_{(y_1,y_2)}(v) &= \tfrac{1}{2}\int_{y_1}^{y_2} \|\partial_y v(\cdot, y)\|_2^2\, dy + \int_{y_1}^{y_2} V(v(\cdot, y)) - b\, dy \\
&\geq \tfrac{1}{2(y_2-y_1)} \int_{\mathbb{R}^N} (\int_{y_1}^{y_2} |\partial_y v(x, y)|\, dy)^2\, dx + \int_{y_1}^{y_2} V(v(\cdot, y)) - b\, dy \\
&\geq \tfrac{1}{2(y_2-y_1)} \|v(\cdot, y_1) - v(\cdot, y_2)\|_2^2 + \int_{y_1}^{y_2} V(v(\cdot, y)) - b\, dy.
\end{aligned}$$

In particular if $V(v(\cdot, y)) \geq b + \nu$ for any $y \in (y_1, y_2)$, then

$$\varphi_{(y_1,y_2)}(v) \geq \tfrac{1}{2(y_2-y_1)} \|v(\cdot, y_1) - v(\cdot, y_2)\|_2^2 + \nu(y_2 - y_1) \geq \sqrt{2\nu}\, \|v(\cdot, y_1) - v(\cdot, y_2)\|_2. \tag{3.2}$$

Remark 3.4 In the sequel we will denote

$$\delta_0 = \delta((b + c)/2, (b + c)/2) := \mathrm{dist}(\mathcal{V}_-^{(b+c)/2}, \mathcal{V}_+^{(b+c)/2}) \quad \text{and} \quad r_0 = \frac{\delta_0}{5}$$

By (3.2) we can plainly prove that $m_b > 0$. Indeed, note that if $v \in \mathcal{X}_b$, since by Lemma 2.10 we have $\delta_0 > 0$, by (3.1), there exist $y_1 < y_2 \in \mathbb{R}$ such that $\|v(\cdot, y_1) - v(\cdot, y_2)\| \geq \delta_0$ and $V(v(\cdot, y)) > (b + c)/2$ for any $y \in (y_1, y_2)$. Then, by (3.2) we obtain $\varphi_{(y_1,y_2)}(u) \geq \sqrt{c - b}\,\delta_0 > 0$. In particular

$$m_b \geq \sqrt{c - b}\,\delta_0.$$

One of the basic properties defining $\mathcal{X}_b$ is the fact that if $v \in \mathcal{X}_b$ then $V(v(\cdot, y)) \geq b$ for a.e. $y \in \mathbb{R}$. Unfortunately this condition is not necessarily preserved by the weak H^1_{loc} convergence. To overcome this difficulty it is important the following Lemma.

Lemma 3.4 *Let $v \in \mathcal{H}$ and $-\infty \leq \sigma < \tau \leq +\infty$ be such that*

i) $V(v(\cdot, y)) > b$ for any $y \in (\sigma, \tau)$

ii) either $\sigma = -\infty$ and $\liminf_{y\to-\infty} \mathrm{dist}(v(\cdot, y), \mathcal{V}_-^b) = 0$ or $\sigma \in \mathbb{R}$ and $v(\cdot, \sigma) \in \mathcal{V}_-^b$

iii) either $\tau = +\infty$ and $\liminf_{y\to+\infty} \mathrm{dist}(v(\cdot, y), \mathcal{V}_+^b) = 0$ or $\tau \in \mathbb{R}$ and $v(\cdot, \tau) \in \mathcal{V}_+^b$

then $\varphi_{(\sigma,\tau)}(v) \geq m_b$. Moreover if $\liminf_{y\to\sigma^+} V(v(\cdot, y)) > b$ or $\liminf_{y\to\tau^-} V(v(\cdot, y)) > b$ then $\varphi_{(\sigma,\tau)}(v) > m_b$.

Proof. We consider the case in which σ, $\tau \in \mathbb{R}$. Similar arguments can be used to prove the statement in the cases $\sigma = -\infty$ or $\tau = +\infty$. We fix two sequences (s_n), $(t_n) \subset (\sigma, \tau)$ such that $s_n \to \sigma$, $t_n \to \tau$ as $n \to +\infty$ and

$$V(v(\cdot, s_n)) \leq \inf_{y\in(\sigma,s_n)} V(v(\cdot, y)) + \tfrac{1}{n} \text{ and } V(v(\cdot, t_n)) \leq \inf_{y\in(t_n,\tau)} V(v(\cdot, y)) + \tfrac{1}{n}. \tag{3.3}$$

Moreover, since by (3.1),we have $\|v(\cdot, s_n) - v(\cdot, \sigma)\|_2 \to 0$ and $\|v(\cdot, t_n) - v(\cdot, \tau)\|_2 \to 0$ as $n \to +\infty$, it is not restrictive to assume that

$$\|v(\cdot, s_n) - v(\cdot, \sigma)\|_2 \leq r_0 \text{ and } \|v(\cdot, t_n) - v(\cdot, \tau)\|_2 \leq r_0 \text{ for any } n \in \mathbb{N} \tag{3.4}$$

For any $n \in \mathbb{N}$, consider the paths in $\mathcal{X}$ defined by

$$\begin{aligned}\gamma_{n,-}(y) &= v(\cdot,\sigma) + \tfrac{y-\sigma}{s_n-\sigma}\left(v(\cdot,s_n) - v(\cdot,\sigma)\right), \quad y \in [\sigma, s_n],\\ \gamma_{n,+}(y) &= v(\cdot,\tau) + \tfrac{\tau-y}{\tau-t_n}\left(v(\cdot,t_n) - v(\cdot,\tau)\right), \quad y \in [t_n, \tau].\end{aligned}$$

Note that, for any $n \in \mathbb{N}$, the paths $\gamma_{n,-}$ and $\gamma_{n,+}$ continuously connect in $\mathcal{X}$ respectively the points $v(\cdot,\sigma)$, $v(\cdot,s_n)$ and $v(\cdot,\tau)$, $v(\cdot,t_n)$. Then, since by $(ii)-(iii)$, $V(v(\cdot,\sigma))$, $V(v(\cdot,\tau)) \le b$ and $V(v(\cdot,s_n))$, $V(v(\cdot,t_n)) > b$, defining for $n \in \mathbb{N}$

$$\begin{aligned}\bar{s}_n &= \inf\{\bar{y} \in [\sigma, s_n] \,/\, V(\gamma_{n,-}(y)) \ge b \text{ for any } y \in [\bar{y}, s_n]\},\\ \bar{t}_n &= \sup\{\bar{y} \in [t_n, \tau] \,/\, V(\gamma_{n,+}(y)) \ge b \text{ for any } y \in [t_n, \bar{y}]\},\end{aligned}$$

by continuity, we have that $V(\gamma_{n,-}(\bar{s}_n)) = b$ and $V(\gamma_{n,+}(\bar{t}_n)) = b$. Moreover, by definition, $V(\gamma_{n,-}(y)) \ge b$ for any $y \in [\bar{s}_n, s_n]$ and $V(\gamma_{n,+}(y)) \ge b$ for any $y \in [t_n, \bar{t}_n]$.
Define, for n, $j \in \mathbb{N}$,

$$w_{n,j}(\cdot,y) = \begin{cases} \gamma_{n,-}(\bar{s}_n) & \text{if } y \le \bar{s}_n,\\ \gamma_{n,-}(y) & \text{if } \bar{s}_n < y \le s_n,\\ v(\cdot,y) & \text{if } s_n < y \le t_j,\\ \gamma_{j,+}(y) & \text{if } t_j < y \le \bar{t}_j,\\ \gamma_{j,+}(\bar{t}_n) & \text{if } \bar{t}_j < y, \end{cases}$$

and note that $w_{n,j} \in \mathcal{X}_b$, and so $\varphi(w_{n,j}) \ge m_b$ for any n, $j \in \mathbb{N}$.
To prove that $\varphi_{(\sigma,\tau)}(v) \ge m_b$,

we estimate the difference $\varphi_{(\sigma,\tau)}(v) - \varphi(w_{n,j})$. To this end, note that, since $\varphi(w_{n,j}) = \varphi_{(\sigma,\tau)}(w_{n,j})$ and since $w_{n,j}(\cdot,y) = v(\cdot,y)$ for any $y \in (s_n, t_j)$, we have

$$\begin{aligned}\varphi_{(\sigma,\tau)}(v) - \varphi(w_{n,j}) = &\int_\sigma^{s_n} \tfrac{1}{2}\left(\|\partial_y v(\cdot,y)\|_2^2 - \|\partial_y w_{n,j}(\cdot,y)\|_2^2\right) + \left(V(v(\cdot,y)) - V(w_{n,j}(\cdot,y))\right) dy\\ &+ \int_{t_j}^{\tau} \tfrac{1}{2}\left(\|\partial_y v(\cdot,y)\|_2^2 - \|\partial_y w_{n,j}(\cdot,y)\|_2^2\right) + \left(V(v(\cdot,y)) - V(w_{n,j}(\cdot,y))\right) dy. \qquad (3.5)\end{aligned}$$

Since $\partial_y w_{n,j}(\cdot,y) = \partial_y \gamma_{n,-}(\cdot,y) = \frac{1}{s_n-\sigma}\left(v(\cdot,s_n) - v(\cdot,\sigma)\right)$ for $y \in (\bar{s}_n, s_n)$ and $\partial_y w_{n,j}(\cdot,y) = 0$ for $y \in (\sigma, \bar{s}_n)$, by (3.1) we recover that

$$\int_\sigma^{s_n} \|\partial_y w_{n,j}(\cdot,y)\|_2^2\, dy \le \tfrac{1}{s_n-\sigma}\|v(\cdot,s_n) - v(\cdot,\sigma)\|_2^2 \le \int_\sigma^{s_n} \|\partial_y v(\cdot,y)\|_2^2\, dy.$$

Analogously, we obtain also that $\int_{t_j}^{\tau} \|\partial_y w_{n,j}(\cdot,y)\|_2^2 dy \le \int_{t_j}^{\tau} \|\partial_y v(\cdot,y)\|_2^2 dy$ and by (3.5) we conclude

$$\varphi_{(\sigma,\tau)}(v) - \varphi(w_{n,j}) \ge \int_\sigma^{s_n} V(v(\cdot,y)) - V(w_{n,j}(\cdot,y))\, dy + \int_{t_j}^{\tau} V(v(\cdot,y)) - V(w_{n,j}(\cdot,y))\, dy. \qquad (3.6)$$

Let us prove that $\liminf_{n\to+\infty} \int_\sigma^{s_n} V(v(\cdot,y)) - V(w_{n,j}(\cdot,y))\, dy \ge 0$ for any $j \in \mathbb{N}$. Since $V(w_{n,j}(\cdot,y)) = V(\gamma_{n,-}(y))$ for any $y \in (\bar{s}_n, s_n)$ and $V(w_{n,j}(\cdot,y)) = b$ for any $y \in (\sigma, \bar{s}_n)$, we have

$$\begin{aligned}\int_\sigma^{s_n} V(v(\cdot,y)) - V(w_{n,j}(\cdot,y))\, dy = &\int_\sigma^{s_n} V(v(\cdot,y)) - V(v(\cdot,s_n))\, dy\\ &+ \int_\sigma^{\bar{s}_n} V(v(\cdot,s_n)) - b\, dy + \int_{\bar{s}_n}^{s_n} V(v(\cdot,s_n)) - V(\gamma_{n,-}(y))\, dy.\end{aligned}$$

We separately estimate the three addenda at the right hand side of the above equality. For the first one that by (3.3)

$$\int_{\sigma}^{s_n} V(v(\cdot,y)) - V(v(\cdot,s_n))\,dy \geq -\tfrac{1}{n}(s_n-\sigma). \tag{3.7}$$

For the second one, we set

$$\ell := \min\{\liminf_{y\to\sigma^+} V(v(\cdot,y)),\, b+1\}$$

and note that $\ell \in [b, b+1]$ and by (3.3) we have in particular $\ell \leq \liminf_{y\to\sigma^+} V(v(\cdot,y)) = \lim_{n\to+\infty} V(v(\cdot,s_n))$ and hence that $V(v(\cdot,s_n)) - b \geq \ell - b + o(1)$ with $o(1)\to 0$ as $n\to+\infty$. Then

$$\int_{\sigma}^{\bar{s}_n} V(v(\cdot,y)) - b\,dy \geq (\ell - b + o(1))\,(\bar{s}_n - \sigma). \tag{3.8}$$

Finally, for the third addendum, setting $4\mu = \ell - V(v(\cdot,\sigma))$, we consider the two alternative cases: $\mu = 0$ or $\mu > 0$.
If $\mu = 0$, since by (ii), $V(v(\cdot,\sigma)) \leq b$ and $\ell \geq b$, we derive that $b = \ell = V(v(\cdot,\sigma))$ and so, by (3.3), that $\lim_{n\to+\infty} V(v(\cdot,s_n)) = V(v(\cdot,\sigma)) = b$. Then by Lemma 2.11 and Remark 2.7 we derive $\bar{s}_n = \sigma$ and $v(\cdot,s_n)\to v(\cdot,\sigma)$ strongly in $\mathcal{X}$. Then, for any $y \in [\bar{s}_n, s_n]$ we have $\|\gamma_{n,-}(y) - v(\cdot,\sigma)\| \leq \|v(\cdot,s_n) - v(\cdot,\sigma)\|$ and by continuity of V we obtain $\sup_{y\in(\bar{s}_n,s_n)} V(v(\cdot,s_n)) - V(\gamma_{n,-}(y)) \to 0$ as $n\to+\infty$. This allows us to conclude that if $\ell = V(v(\cdot,\sigma))$ we have

$$\int_{\bar{s}_n}^{s_n} V(v(\cdot,s_n)) - V(\gamma_{n,-}(y))\,dy \geq o(1)(s_n - \bar{s}_n) \text{ as } n\to+\infty. \tag{3.9}$$

In the second case, i.e. $\mu > 0$, we have that necessarily $\liminf_{n\to+\infty} \|\nabla(v(\cdot,s_n) - v(\cdot,s))\| \geq 8\mu_0$ for a certain $\mu_0 > 0$. Then setting $\sigma_n(y) = \frac{y-\sigma}{s_n-\sigma}$ and $v_n = v(\cdot,s_n) - v(\cdot,\sigma)$, by Lemma 2.11, we obtain that for n sufficiently large and $y \in (\bar{s}_n, s_n)$ we have

$$V(v(\cdot,s_n)) - V(\gamma_{n,-}(y)) = V(v(\cdot,\sigma)+v_n) - V(v(\cdot,\sigma)+\sigma_n(y)v_n) \geq \mu_0(1-\sigma_n(y)) = \mu_0\tfrac{s_n-y}{s_n-\sigma}.$$

Then

$$\int_{\bar{s}_n}^{s_n} V(v(\cdot,s_n)) - V(\gamma_{n,-}(y))\,dy \geq \mu_0\int_{\bar{s}_n}^{s_n} \tfrac{s_n-y}{s_n-\sigma}\,dy = \tfrac{\mu_0}{2}\tfrac{s_n-\bar{s}_n}{s_n-\sigma}(s_n-\bar{s}_n). \tag{3.10}$$

By (3.9) and (3.10) we obtain

$$\int_{\bar{s}_n}^{s_n} V(v(\cdot,s_n)) - V(\gamma_{n,-}(y))\,dy \geq (\tfrac{\mu_0}{2}\tfrac{s_n-\bar{s}_n}{s_n-\sigma} + o(1))(s_n-\bar{s}_n). \tag{3.11}$$

Gathering (3.7), (3.8), (3.11), we conclude that if n is sufficiently large then, for any $j\in\mathbb{N}$,

$$\begin{aligned}\int_{\sigma}^{s_n} V(v(\cdot,y)) - V(w_{n,j}(\cdot,y))\,dy &\geq -\tfrac{1}{n}(s_n-\sigma) + (\ell - c + o(1))(\bar{s}_n-\sigma)+\\ &\quad + (\tfrac{\mu_0}{2}\tfrac{s_n-\bar{s}_n}{s_n-\sigma} + o(1))(s_n-\bar{s}_n).\end{aligned} \tag{3.12}$$

and then, $\liminf_{n\to+\infty}\int_{\sigma}^{s_n} V(v(\cdot,y)) - V(w_{n,j}(\cdot,y))\,dy \geq 0$. In a symmetric way, we can prove that $\liminf_{j\to+\infty}\int_{t_j}^{\tau} V(v(\cdot,y)) - V(w_{n,j}(\cdot,y))\,dy \geq 0$ for every $n\in\mathbb{N}$. Then, since $\varphi(w_{n,j}) \geq m_b$, by (3.6) we conclude $\varphi_{(\sigma,\tau)}(v) \geq m_b$.

Let us finally prove that if $\liminf_{y\to\sigma^+} V(v(\cdot,y)) > b$ then $\varphi_{(\sigma,\tau)}(v) > m_b$.
Considering ℓ and μ_0 defined as above we have $\ell > b$ and $\mu_0 > 0$ and hence $2\tilde{\mu} := \min\{\ell - b, \frac{\mu_0}{2}\} > 0$. Since $(\bar{s}_n - \sigma)^2 + (s_n - \bar{s}_n)^2 \geq \frac{1}{2}(s_n - s)^2$, by (3.12) we obtain that for n large and $j \in \mathbb{N}$

$$\begin{aligned}\int_\sigma^{s_n} V(v(\cdot,y)) - V(w_{n,j}(\cdot,y))\,dy &\geq o(1)(s_n-\sigma) + \tilde{\mu}\left[(\bar{s}_n-\sigma) + \tfrac{(s_n-\bar{s}_n)^2}{s_n-\sigma}\right] \\ &\geq o(1)(s_n-\sigma) + \tilde{\mu}\left[\tfrac{(\bar{s}_n-\sigma)^2+(s_n-\bar{s}_n)^2}{(s_n-\sigma)^2}\right](s_n-\sigma) \\ &\geq \tfrac{\tilde{\mu}}{4}(s_n-\sigma).\end{aligned}$$

Then, by (3.6), since $\liminf_{j\to+\infty}\int_{t_j}^{\tau} V(v(\cdot,y)) - V(w_{n,j}(\cdot,y))\,dy \geq 0$ for every $n \in \mathbb{N}$, we recover that for n sufficiently large

$$\varphi_{(\sigma,\tau)}(v) - m_b \geq \liminf_{j\to+\infty}\,[\varphi_{(\sigma,\tau)}(v) - \varphi(w_{n,j})] \geq \tfrac{\tilde{\mu}}{4}(s_n - s) > 0.$$

Simmetricaly we can prove that if $\liminf_{y\to\tau^-} V(v(\cdot,y)) > b$ then $\varphi_{(\sigma,\tau)}(v) > m_b$. $\square$

3.2 Estimates near the boundary of $\mathcal{V}_-^b$ and $\mathcal{V}_+^b$

To study coercivity property of φ we first establish some technical local results. We define the constants (depending on b)

$$\beta = b + \tfrac{c-b}{4}, \quad \text{and } \Lambda_0 = \sqrt{\tfrac{c-b}{2}}\,\tfrac{r_0}{4} \tag{3.13}$$

where δ_0 and r_0 are definined in Remark 3.4, noting that

$$\mathrm{dist}(\mathcal{V}_-^b, \mathcal{V}_+^b) \geq \mathrm{dist}(\mathcal{V}_-^\beta, \mathcal{V}_+^\beta) \geq 5r_0. \tag{3.14}$$

Given $u_0 \in \mathcal{X}$ we denote

$$\begin{aligned}\mathcal{X}_{b,u_0}^- &= \{v \in \mathcal{H} \,/\, v(\cdot,0) = u_0,\ \inf_{(-\infty,0)} V(v(\cdot,y)) \geq b,\ \liminf_{y\to-\infty}\mathrm{dist}(v(\cdot,y),\mathcal{V}_-^b) = 0\},\\ \mathcal{X}_{b,u_0}^+ &= \{v \in \mathcal{H} \,/\, v(\cdot,0) = u_0,\ \inf_{(0,+\infty)} V(v(\cdot,y)) \geq b,\ \liminf_{y\to+\infty}\mathrm{dist}(v(\cdot,y),\mathcal{V}_+^b) = 0\}.\end{aligned}$$

Next Lemma establishes that if $v \in \mathcal{X}_{b,u_0}^+$ (resp. $\mathcal{X}_{b,u_0}^-$) is such that $\varphi_{(0,+\infty)}(v)$ (resp. $\varphi_{(-\infty,0)}(v)$) is sufficiently small, then the trajectory $y \to v(\cdot,y)$ remains close to the set $\mathcal{V}_+^\beta$ (resp. $\mathcal{V}_-^\beta$) with respect to the $L^2(\mathbb{R}^N)$ metric.

Lemma 3.5 *If $u_0 \in \mathcal{X}$, $V(u_0) \geq b$, $v \in \mathcal{X}_{b,u_0}^+$ (resp. $v \in \mathcal{X}_{b,u_0}^-$) and $\varphi_{(0,+\infty)}(v) \leq \Lambda_0$ (resp. $\varphi_{(-\infty,0)}(v) \leq \Lambda_0$) then $\mathrm{dist}(v(\cdot,y),\mathcal{V}_+^\beta) \leq r_0$ for every $y \in [0,+\infty)$ (resp. $\mathrm{dist}(v(\cdot,y),\mathcal{V}_+^\beta) \leq r_0$ for every $y \in (-\infty,0]$).*

Proof. By (3.1) the function $y \in [0,+\infty) \mapsto v(\cdot,y) \in L^2(\mathbb{R}^n)$ is continuous. Hence, using Remark 2.6, the map $y \in [0,+\infty) \mapsto \mathrm{dist}(v(\cdot,y),\mathcal{V}_+^\beta)$ is continuous too. If, by contradiction, $y_0 \geq 0$ is such that $\mathrm{dist}(v(\cdot,y_0),\mathcal{V}_+^\beta) > r_0$, since $\liminf_{y\to+\infty}\mathrm{dist}(v(\cdot,y),\mathcal{V}_+^b) = 0$, by continuity there exists an interval $(y_1,y_2) \subset \mathbb{R}$ such that $0 < \mathrm{dist}(v(\cdot,y),\mathcal{V}_+^\beta) < r_0$ for any $y \in (y_1,y_2)$ and $\|v(\cdot,y_1) - v(\cdot,y_2)\|_2 \geq r_0/2$. By (3.14) we derive $v(\cdot,y) \notin \mathcal{V}_+^\beta \cup \mathcal{V}_-^\beta$ and so $V(v(\cdot,y)) - b \geq \beta - b = (c-b)/4$ for all $y \in (y_1,y_2)$. By (3.2) we conclude

$$\Lambda_0 \geq \varphi_{(0,+\infty)}(v) \geq \varphi_{(y_1,y_2)}(v) \geq \sqrt{\tfrac{c-b}{2}}\|v(\cdot,y_1) - v(\cdot,y_2)\|_2 \geq \sqrt{\tfrac{c-b}{2}}\,\tfrac{r_0}{2} = 2\Lambda_0,$$

a contradiction which proves the Lemma. Analogous is the proof in the case $v \in \mathcal{X}_{b,u_0}^-$. $\square$

Clearly the infimum value of $\varphi_{(0,+\infty)}$ on $\mathcal{X}^+_{b,u_0}$ is close to 0 as $\mathrm{dist}(u_0,\mathcal{V}^b)$ is small. Next result displays a test function $w^+_{u_0}\in\mathcal{X}^+_{b,u_0}$ which gives us more precise information.

Lemma 3.6 *Let $b\in[0,c)$, then there exists $C_+(b)>0$ such that for every $u_0\in\mathcal{V}^\beta_+\setminus\mathcal{V}^b_+$ there exists $w^+_{u_0}\in\mathcal{X}^+_{b,u_0}$ such that*

$$\sup_{y>0}\|w^+_{u_0}(\cdot,y)-u_0\|_2\le\tfrac{1}{\nu^+(\beta)}(V(u_0)-b)\quad\text{and}\quad\varphi_{(0,+\infty)}(w^+_{u_0})\le C_+(b)(V(u_0)-b)^{3/2}.$$

Proof. Note that, since $u_0\in\mathcal{V}^\beta_+$, by Lemma 2.3, we have $V'(u_0)u_0<0$ and there exists a unique $s_0\in(1,+\infty)$ such that $V(su_0)>b$ for any $s\in[1,s_0)$ and $V(s_0u_0)=b$. Moreover $\frac{d}{ds}V(su_0)=s(V'(u_0)u_0+\int_{\mathbb{R}^N}f(u_0)u_0-\frac{1}{s}f(su_0)u_0\,dx)$ and since, by (1.6), $\int_{\mathbb{R}^N}f(u_0)u_0-\frac{1}{s}f(su_0)u_0\,dx\le0$ for any $s\ge1$, we deduce that $\frac{d}{ds}V(su_0)\le sV'(u_0)u_0$ for any $s\ge1$. Integrating this last inequality on the interval $[1,s_0]$, we obtain $V(s_0u_0)\le V(u_0)+\frac12(s_0^2-1)V'(u_0)u_0$ and so the estimate $s_0-1\le\frac{V(u_0)-b}{|V'(u_0)u_0|}$. We define

$$w^+_{u_0}(x,y)=\begin{cases}u_0(x) & y\le0\\ (1+\frac{y^2}{2})u_0 & y\in(0,\sqrt{2(s_0-1)})\\ s_0u_0 & y\ge\sqrt{2(s_0-1)}\end{cases}$$

notiing that $w^+_{u_0}\in\mathcal{X}^+_{b,u_0}$ and $\sup_{y\ge0}\|w^+_{u_0}(\cdot,y)-u_0\|_2=(s_0-1)\|u_0\|_2\le\frac{V(u_0)-b}{|V'(u_0)u_0|}\|u_0\|_2$. Moreover, since $s_0-1\le\frac{V(u_0)-b}{|V'(u_0)u_0|}$, we get

$$\begin{aligned}\varphi_{(-\infty,0)}(w^+_{u_0})&=\int_0^{\sqrt{2(s_0-1)}}\tfrac12\|\partial_y(1+\tfrac{y^2}{2})u_0(\cdot)\|_2^2\,dy+\int_0^{\sqrt{2(s_0-1)}}V((1+\tfrac{y^2}{2})u_0(\cdot))-b\,dy\\&\le\int_0^{\sqrt{2(s_0-1)}}\tfrac12y^2\|u_0\|_2^2\,dy+\int_0^{\sqrt{2(s_0-1)}}V(u_0)-b\,dy\\&=\sqrt{2(s_0-1)}(\tfrac{(s_0-1)}{3}\|u_0\|_2^2+(V(u_0)-b))\\&\le\sqrt{\tfrac{2}{|V'(u_0)u_0|}}(\tfrac{1}{3|V'(u_0)u_0|}\|u_0\|_2^2+1)(V(u_0)-b)^{3/2}.\end{aligned}$$

By Lemma 2.6 we know that $|V'(u_0)u_0|\ge\nu^+(\beta)\max\{1,\|u_0\|_2^2\}$ and the Lemma follows considering $C_+(b)=\sqrt{\frac{2}{\nu^+(\beta)}}(\frac{1}{3\nu^+(\beta)}+1)$. ☐

For any $b\in[0,c)$ we fix $\beta_+\in(b,\beta]$ such that the following inequalities hold true:

$$\tfrac{\beta_+-b}{\nu^+(\beta)}<\tfrac12,\quad\max\{1,C_+(b)\}(\beta_+-b)^{1/4}<\tfrac14,\quad C_+(b)(\beta_+-b)^{3/2}\le\Lambda_0.\tag{3.15}$$

Next Gronwall type result will play an important role together with Lemma 3.6.

Lemma 3.7 *Assume that $u_0\in\mathcal{V}^{\beta_+}_+\setminus\mathcal{V}^b_+$ and $v\in\mathcal{X}^+_{b,u_0}$ are such that*

$$\text{if } y\in[0,1)\text{ is such that }V(\bar v(\cdot,y))\le\beta_+\text{ then }\varphi_{(y,+\infty)}(\bar v)\le C_+(b)(V(\bar v(\cdot,y))-b)^{3/2}.\tag{3.16}$$

Then there exists $\bar y\in(0,1)$ such that $V(v(\cdot,\bar y))=b$, $v(\cdot,\bar y)\in\mathcal{V}^b_+$ and $v(\cdot,y)=v(\cdot,\bar y)$ for every $y\in[\bar y,+\infty)$.

Proof. We first note that, since $u_0\in\mathcal{V}^{\beta_+}_+\setminus\mathcal{V}^b_+$ and $v\in\mathcal{X}^+_{b,u_0}$ we have $V(v(\cdot,0))=V(u_0)\le\beta_+$ and hence, by (3.16) and (3.15), we have $\varphi_{(0,+\infty)}(v)\le C_+(b)(V(u_0)-b)^{3/2}\le\Lambda_0$. By

Lemma 3.5 we then deduce that $\mathrm{dist}(v(\cdot,y),\mathcal{V}_+^{\beta}) \le r_0$ for any $y>0$ and, by the definition of r_0, we obtain that $v(\cdot,y)\notin\mathcal{V}_-^{\beta_+}$ for any $y>0$. In particular, if $y>0$ and $V(v(\cdot,y))\le\beta_+$ then $v(\cdot,y)\in\mathcal{V}_+^{\beta_+}$.
We claim that there exists a sequence $(\zeta_n)\subset[0,\frac{1}{2})$ such that

$$\zeta_{n-1}<\zeta_n\le\zeta_{n-1}+(\tfrac{\beta_+-b}{4^{2(n-1)}})^{1/4}<\tfrac{1}{2} \text{ and } V(v(\cdot,\zeta_n))-b\le\tfrac{\beta_+-b}{4^n},\quad \forall n\in\mathbb{N}. \tag{3.17}$$

Indeed, defining $\zeta_0=0$ by (3.15) and (3.16) we have that for any $\zeta>\zeta_0$

$$\begin{aligned}\int_{\zeta_0}^{\zeta} V(v(\cdot,s))-b\,ds &\le \varphi_{(\zeta_0,+\infty)}(v)\le C_+(b)(V(v(\cdot,\zeta_0))-b)^{3/2}\\ &\le C_+(b)(\beta_+-b)^{1/4}(\beta_+-b)(\beta_+-b)^{1/4}\le\tfrac14(\beta_+-b)(\beta_+-b)^{1/4},\end{aligned}$$

and so

$$\exists\,\zeta_1\in(\zeta_0,\zeta_0+(\beta_+-b)^{1/4}) \text{ such that } V(\bar v(\cdot,\zeta_1))-b\le\tfrac{\beta_+-b}{4}, \tag{3.18}$$

Note that, by (3.15), $\zeta_0+(\beta_+-b)^{1/4}<\zeta_0+\frac14<\frac12$ and so $\zeta_1\in(0,\frac12)$.
Now, if ζ_n verifies (3.17) by (3.16) we obtain that for any $\zeta>\zeta_n$

$$\begin{aligned}\int_{\zeta_n}^{\zeta} V(v(\cdot,s))-b\,ds &\le \varphi_{(\zeta_n,+\infty)}(v)\le C_+(b)(V(v(\cdot,\zeta_n))-b)^{3/2}\\ &\le C_+(b)(\beta_+-b)^{1/4}(\tfrac{\beta_+-b}{4^n})(\tfrac{\beta_+-b}{4^{2n}})^{1/4}<\tfrac{\beta_+-b}{4^{n+1}}(\tfrac{\beta_+-b}{4^{2n}})^{1/4},\end{aligned}$$

implying that

$$\exists\,\zeta_{n+1}\in(\zeta_n,\zeta_n+(\tfrac{\beta_+-b}{4^{2n}})^{1/4}) \text{ such that } V(v(\cdot,\zeta_{n+1}))-b\le\tfrac{\beta_+-b}{4^{n+1}},$$

with, by (3.15),

$$\zeta_{n+1}<\sum_{j=0}^{n}(\tfrac{\beta_+-b}{4^{2j}})^{1/4}=(\beta_+-b)^{1/4}\sum_{j=0}^{+\infty}\tfrac{1}{2^j}<\tfrac12.$$

Then, by induction, (3.17) holds true for any $n\in\mathbb{N}$.
Now, note that by (3.17) we have $\zeta_n\to\bar y\in(0,\frac12]$ as $n\to+\infty$. Moreover, since $v\in\mathcal{X}_{b,u_0}$ there result $V(v(\cdot,\zeta_n))\ge b$ for all $n\in\mathbb{N}$ and hence, by (3.17), $V(v(\cdot,\zeta_n))\to b$. Then, by Lemma 3.1, we deduce $V(v(\cdot,\bar y))=b$. Moreover, by (3.1), $v(\cdot,\zeta_n)\to v(\cdot,\bar y)$ in $L^2(\mathbb{R}^N)$. Then we can conclude that $v(\cdot,\bar y)\in\mathcal{V}_+^b$ and hence, using (3.16), that $\varphi_{(\bar y,+\infty)}(v)\le C_+(b)(V(v(\cdot,\bar y))-b)^{3/2}=0$, which implies $v(\cdot,y)=v(\cdot,\bar y)$ for every $y\ge\bar y$. □

Lemma 3.7 and Lemma 3.6 have in particular the following consequence which will be a key tool in constructing minimizing sequences for φ with suitable compactness properties.

Lemma 3.8 *Let $b\in[0,c)$ then, for every $u_0\in\mathcal{V}_+^{\beta_+}\setminus\mathcal{V}_+^b$ and $v\in\mathcal{X}_{b,u_0}^+$ there exists $\tilde v\in\mathcal{X}_{b,u_0}^+$ such that*

$$\sup_{y\in(0,+\infty)}\|\tilde v(\cdot,y)-u_0\|_2\le 1 \text{ and } \varphi_{(0,+\infty)}(\tilde v)\le\min\{\Lambda_0,\varphi_{(0,+\infty)}(v)\}.$$

Proof. Note that, by Lemma 3.6 and (3.15), we have in that if $u_0\in\mathcal{V}_+^{\beta_+}\setminus\mathcal{V}_+^b$ then $\varphi_{(0,+\infty)}(w_{u_0}^+)\le\Lambda_0$ and $\|w_{u_0}^+(\cdot,y)-u_0\|\le\frac12$ for any $y>0$. In particular if $u_0\in\mathcal{V}_+^{\beta_+}\setminus\mathcal{V}_+^b$ and $v\in\mathcal{X}_{b,u_0}^+$ are such that $\varphi_{(0,+\infty)}(v)>\Lambda_0$ then the statement of the Lemma holds true with $\tilde v=w_{u_0}^+$.

To prove the Lemma we argue by contradiction assuming that there exist $u_0 \in \mathcal{V}_+^{\beta_+} \setminus \mathcal{V}_+^b$ and $v \in \mathcal{X}^+_{b,u_0}$ with $\varphi_{(0,+\infty)}(v) \le \Lambda_0$ such that

$$\varphi_{(0,+\infty)}(\tilde v) > \varphi_{(0,+\infty)}(v) \text{ for every } \tilde v \in \mathcal{X}^+_{b,u_0} \text{ such that } \sup_{y\in(0,+\infty)} \|\tilde v(\cdot,y) - u_0\|_2 \le 1. \quad (3.19)$$

By (3.19) we have $\sup_{y\in(0,+\infty)} \|v(\cdot,y)-u_0\|_2 > 1$ and since $v(\cdot,0) = u_0$, by (3.1) we recover that

$$\exists\, y_0 > 0 \text{ such that } \|v(\cdot,y_0) - u_0\|_2 = \tfrac{1}{2} \text{ and } \|v(\cdot,y)-u_0\|_2 < \tfrac{1}{2} \text{ for any } y \in [0,y_0). \quad (3.20)$$

As already noted in the proof of the previous Lemma, by Lemma 3.5, since $\varphi_{(0,+\infty)}(v) \le \Lambda_0$, we have that if $y > 0$ and $V(v(\cdot,y)) \le \beta_+$ then $v(\cdot,y) \in \mathcal{V}_+^{\beta_+}$. We deduce that

$$\text{if } \tilde y \in [0,y_0) \text{ and } V(v(\cdot,\tilde y)) \le \beta_+ \text{ then } \varphi_{(\tilde y,+\infty)}(v) \le C_+(b)(V(v(\cdot,\tilde y)) - b)^{3/2}. \quad (3.21)$$

Indeed, considering the function

$$\tilde v(\cdot,y) = \begin{cases} v(\cdot,y) & 0 \le y < \tilde y \\ w^+_{v(\cdot,\tilde y)}(\cdot, y-\tilde y) & y \ge \tilde y, \end{cases}$$

we have $\tilde v \in \mathcal{X}^+_{b,u_0}$. Now note that for every $y \in [0,\tilde y) \subset [0,y_0)$, by definition of y_0 we have $\|\tilde v(\cdot,y)-u_0\|_2 = \|v(\cdot,y)-u_0\|_2 < \frac{1}{2}$ while if $y \ge \tilde y$, then by Lemma 3.6 and (3.15)

$$\|\tilde v(\cdot,y) - u_0\|_2 = \|w^+_{v(\cdot,\tilde y)}(\cdot,y-\tilde y) - u_0\|_2 \le \frac{\beta_+ - b}{\nu^+(\beta_+)} < \frac{1}{2}.$$

Hence we recover that $\sup_{y>0} \|\tilde v(\cdot,y)-u_0\|_2 \le 1$. Then, by (3.19) we obtain $\varphi_{(0,+\infty)}(v) < \varphi_{(0,+\infty)}(\tilde v) \le \varphi_{(\tilde y,+\infty)}(\tilde v) = \varphi_{(0,+\infty)}(w^+_{v(\cdot,\tilde y)})$ and (3.21) follows by Lemma 3.6.
We now note that, by Remark 3.3 we have $\varphi_{(0,y_0)}(v) \ge \frac{1}{2y_0}\|v(\cdot,y_0)-u_0\|_2^2 = \frac{1}{8y_0}$ and so, by (3.15) and (3.21), we deduce $y_0 \ge \frac{1}{8C_+(b)(\beta_+-b)^{3/2}} > 1$. Then, by (3.21) and Lemma 3.7, we derive that there exists $\bar y \in (0,1)$ such that $v(\cdot,\bar y) \in \mathcal{V}_+^b$ and $v(\cdot,y) = v(\cdot,\bar y)$ for any $y \ge \bar y$. Hence, using (3.20), we obtain $1 < \sup_{y\in(0,+\infty)} \|v(\cdot,y)-u_0\|_2 = \sup_{y\in(0,\bar y]} \|\, v(\cdot,y)-u_0\|_2 \le \sup_{y\in(0,y_0]} \|v(\cdot,y)-u_0\|_2 = \frac{1}{2}$, a contradiction which proves the Lemma. □

The following Lemma is an analogous of Lemma 3.6 for $\mathcal{X}^-_{b,u_0}$ when $b > 0$. We omit the proof since it is bases on an argument symmetric to the one used proving Lemma 3.6, using Lemma 2.7 instead of Lemma 2.6.

Lemma 3.9 *Let $b \in (0,c)$, then there exists $C_-(b) > 0$ such that for any $u_0 \in \mathcal{V}_-^\beta \setminus \mathcal{V}_-^b$ there exists $w^-_{u_0} \in \mathcal{X}^-_{b,u_0}$ such that*

$$\varphi_{(-\infty,0)}(w^-_{u_0}) \le C_-(b)(V(u_0)-b)^{3/2}.$$

For any $b \in (0,c)$ we fix $\beta_- \in (b,\beta]$ such that the following inequalities hold true:

$$\max\{1, C_-(b)\}(\beta_- - b)^{1/4} < \tfrac{1}{4} \quad \text{and} \quad C_-(b)(\beta_- - b)^{3/2} \le \Lambda_0. \quad (3.22)$$

Analogously to Lemma 3.7 we can prove

Lemma 3.10 *Let $b \in (0, c)$ and assume that $u_0 \in \mathcal{V}_-^{\beta_-} \setminus \mathcal{V}_-^b$ and $v \in \mathcal{X}_{b,u_0}^-$ are such that*

$$\text{if } y \in (-1, 0] \text{ is such that } V(v(\cdot, y)) \leq \beta_- \text{ then } \varphi_{(-\infty,y)}(v) \leq C_-(b)(V(v(\cdot, y)) - b)^{3/2}. \tag{3.23}$$

Then, there exists $\bar{y} \in (-1, 0)$ such that $V(v(\cdot, \bar{y})) = b$, $v(\cdot, \bar{y}) \in \mathcal{V}_-^b$ and $v(\cdot, y) = v(\cdot, \bar{y})$ for any $y \in (-\infty, \bar{y}]$.

The situation is slightly different when $b = 0$.

Lemma 3.11 *If $b = 0$ there exists $\beta_0 \in (0, \frac{c}{4})$ such that for any $u_0 \in \mathcal{V}_-^{\beta_0} \setminus \{0\}$ there exists $w_{u_0}^- \in \mathcal{X}_{b,u_0}^-$ such that $\varphi_{(-\infty,0)}(w_{u_0}^-) \leq 3V(u_0)$.*

Proof. If $u_0 \in \mathcal{V}_-^{\beta_0}$ for some $\beta_0 \in (0, \frac{c}{4})$ we set

$$w_{u_0}^-(x, y) = \begin{cases} u_0(x) & y \geq 0 \\ (1 + y)u_0(x) & y \in (-1, 0) \\ 0 & y \leq -1 \end{cases}$$

noting that $w_{u_0}^- \in \mathcal{X}_{0,u_0}^-$ and $\varphi_{(-\infty,0)}(w_{u_0}^-) \leq \int_{-1}^0 \frac{1}{2}\|u_0\|_2^2 + V(u_0)\, dy \leq \frac{1}{2}\|u_0\|_2^2 + V(u_0)$. By Remark 2.5 and Lemma 2.2, if β_0 is sufficiently small, we obtain $\|u_0\|_2^2 \leq 4V(u_0)$ and so $\varphi_{(-\infty,0)}(w_{u_0}^-) \leq 3V(u_0)$. □

Remark 3.5 Eventually taking β_0 smaller, we can assume that $\varphi_{(-\infty,0)}(w_{u_0}^-) \leq \Lambda_0$ for $u_0 \in \mathcal{V}_-^{\beta_0}$

3.3 Minimizing sequences and their limit points

The local results that we have described in the previous section, allow us to produce a minimizing sequence of φ on $\mathcal{X}_b$ with suitable compactness properties.

Lemma 3.12 *For every $b \in [0, c)$ there exists $L_0 > 0$, $\bar{C} > 0$ and $(v_n) \subset \mathcal{X}_b$ such that $\varphi(v_n) \to m_b$ and*

i) $\operatorname{dist}(v_n(\cdot, y), \mathcal{V}_-^\beta) \leq r_0$ *for any $y \leq 0$ and $n \in \mathbb{N}$,*

ii) $\operatorname{dist}(v_n(\cdot, y), \mathcal{V}_+^\beta) \leq r_0$ *for any $y \geq L_0$ and $n \in \mathbb{N}$,*

iii) $\|v_n(\cdot, y)\|_2 \leq \bar{C}$ *for any $y \in \mathbb{R}$ and $n \in \mathbb{N}$*

iv) for every bounded interval $(y_1, y_2) \subset \mathbb{R}$ there exists $\hat{C} > 0$, depending only on $y_2 - y_1$, such that $\|v_n\|_{H^1(S_{(y_1,y_2)})} \leq \hat{C}$.

Proof. Let $b \in [0, c)$ and $(w_n) \subset \mathcal{X}_b$ be such that $\varphi(w_n) \leq m_b + 1$ for any $n \in \mathbb{N}$ and $\varphi(w_n) \to m_b$. We denote $\beta^* = \min\{\beta_-, \beta_+\}$. Let $s_n = \sup\{y \in \mathbb{R} \mid \varphi_{(-\infty,y)}(w_n) \leq \Lambda_0\}$ and note that by Remark 3.4, since $\Lambda_0 < m_b \leq \varphi(w_n)$, we have $s_n \in \mathbb{R}$ and $\varphi_{(-\infty,s_n)}(w_n) = \Lambda_0$. Since $w_n(\cdot, \cdot + s_n) \in \mathcal{X}_{b,w_n(\cdot,s_n)}^-$ and $\varphi_{(-\infty,0)}(w_n(\cdot, \cdot + s_n)) = \Lambda_0$, by Lemma 3.5 we derive that $\operatorname{dist}(w_n(\cdot, y + s_n), \mathcal{V}_-^\beta) \leq r_0$ for any $y \leq 0$ and so, by (3.14), $\operatorname{dist}(w_n(\cdot, y), \mathcal{V}_+^{b^*}) \geq 4r_0$ for any $y \leq s_n$. We conclude that if $y \leq s_n$ and $V(w_n(\cdot, y)) \leq b^*$ then $w_n(\cdot, y) \in \mathcal{V}_-^{b^*}$. A symmetric argument shows that there exists $t_n > s_n$ such that if $y \geq t_n$ and $V(w_n(\cdot, y)) \leq b^*$ then $w_n(\cdot, y) \in \mathcal{V}_+^{b^*}$. Define now

$$y_n^- = \sup\{y \leq t_n \mid w_n(\cdot, y) \in \mathcal{V}_-^{b^*}\} \text{ and } y_n^+ = \inf\{y \geq y_n^- \mid w_n(\cdot, y) \in \mathcal{V}_+^{b^*}\}.$$

Since $\liminf_{y\to\pm\infty} V(w_n(\cdot,y)) = b > \beta^*$ we deduce that $y_n^-, y_n^+ \in \mathbb{R}$.
Using Remarks 2.5 and 2.6 we also recognize that $w_n(\cdot,y_n^-) \in \mathcal{V}_-^{b^*}$ and $w_n(\cdot,y_n^+) \in \mathcal{V}_+^{b^*}$. Since the function $y \mapsto w_n(\cdot,y)$ is continuous with respect to the $L^2(\mathbb{R}^N)$ metric and $\mathrm{dist}(\mathcal{V}_-^{\beta^*},\mathcal{V}_+^{\beta^*}) \geq 5r_0$ we deduce $y_n^- < y_n^+$. Moreover $V(w_n(\cdot,y)) > \beta^*$ for any $y \in (y_n^-, y_n^+)$ and $\|w_n(\cdot,y_n^+) - w_n(\cdot,y_n^-)\|_2 \geq 5r_0$. By (3.2) we derive

$$y_n^+ - y_n^- \leq \frac{\varphi_{(y_n^-,y_n^+)}(w_n)}{\beta^*-b} \leq \frac{m_b+1}{\beta^*-b} := L_0 \text{ and } \sup_{y\in(y_n^-,y_n^+]} \|w_n(\cdot,y) - w_n(\cdot,y_n^-)\|_2 \leq \frac{m_b+1}{\sqrt{2(\beta^*-b)}}. \tag{3.24}$$

We now claim that, eventually modifying the function w_n on the set $\mathbb{R}^N \times [(-\infty,y_n^-) \cup (y_n^+,+\infty)]$, w_n satisfies

(I) $\varphi_{(-\infty,y_n^-)}(w_n) \leq \Lambda_0$,

(II) $\varphi_{(y_n^+,+\infty)}(w_n) \leq \Lambda_0$ and $\|w_n(x,y) - w_n(x,y_n^+)\|_2 \leq 1$ for any $y \geq y_n^+$.

Indeed, if (I) is not satisfied, since $w_n(\cdot,y_n^-) \in \mathcal{V}_-^{\beta-}$, we can consider the new function

$$w_n^*(\cdot,y) = \begin{cases} w^-_{w_n(\cdot,y_n^-)}(\cdot, y-y_n^-) & \text{if } y \leq y_n^- \\ w_n(\cdot,y) & \text{if } y > y_n^-. \end{cases}$$

nothing that $w_n^* \in \mathcal{X}_b$, $\varphi(w_n^*) \leq \varphi(w_n)$ and w_n^* satisfies (I) by Lemma 3.9, Lemma 3.11, (3.22) and Remark 3.5 .
Now, assuming that (I) is verified, if (II) is not satisfied, since $w_n(\cdot,y_n^+) \in \mathcal{V}_+^{\beta+}$ and $w_n(\cdot,\cdot+y_n^+) \in \mathcal{X}^+_{b,w_n(\cdot,y_n^+)}$, by Lemma 3.8 there exists a function $\tilde{w}_n \in \mathcal{X}^+_{b,w_n(\cdot,y_n^+)}$ such that $\varphi_{(y_n^+,+\infty)}(\tilde{w}_n(\cdot,\cdot-y_n^+)) \leq \min\{\Lambda_0, \varphi_{(y_n^+,+\infty)}(w_n)\}$ and $\|\tilde{w}_n(\cdot,y-y_n^+) - w_n(x,y_n^+)\|_2 \leq 1$ for any $y \geq y_n^+$. Then considering

$$w_n^{**}(\cdot,y) = \begin{cases} \tilde{w}_n(\cdot,y-y_n^+) & \text{if } y \geq y_n^+ \\ w_n(\cdot,y) & \text{if } y < y_n^+. \end{cases}$$

we recognize that $w_n^{**} \in \mathcal{X}_b$, $\varphi(w_n^{**}) \leq \varphi(w_n)$ and w_n^{**} satisfies (I) and (II). Hence, eventually modifying w_n as indicated above our claim follows.
We finally set $v_n = w_n(\cdot,\cdot+y_n^-)$ obtaing that $v_n \in \mathcal{X}_b$ and $\varphi(v_n) = \varphi(w_n) \to m_b$. Moreover, by (I) we have $\varphi_{(-\infty,0)}(v_n) = \varphi_{(-\infty,y_n^-)}(w_n) \leq \Lambda_0$ and (i) follows by Lemma 3.5.
Since by (3.24) we have $y_n^+ - y_n^- \leq L_0$, by (II) we have $\varphi_{(L_0,+\infty)}(v_n) = \varphi_{(L_0+y_n^-,+\infty)}(w_n) \leq \varphi_{(y_n^+,+\infty)}(w_n) \leq \Lambda_0$ and (ii) follows by Lemma 3.5.
To prove (iii) we first note that by Remark 2.5 we have $\|u\|^2 \leq \frac{2\mu}{\mu-2}\beta$ for any $u \in \mathcal{V}_-^{\beta}$. Then, by (i) we recover that $\|v_n(\cdot,y)\|_2^2 \leq \frac{2\mu}{\mu-2}\beta + r_0$ for any $y \leq 0$. Since $v_n(\cdot,0) = w_n(\cdot,y_n^-) \in \mathcal{V}_-^{\beta}$, by (3.24) we obtain moreover $\|v_n(\cdot,y)\|_2^2 \leq \|v_n(\cdot,0)\|_2 + \|v_n(\cdot,y) - v_n(\cdot,0)\|_2 \leq \frac{2\mu}{\mu-2}\beta + \frac{m_b+1}{\sqrt{2(b^*-b)}}$ for any $y \in (0, y_n^+ - y_n^-]$. Finally, by (II), we have $\|v_n(\cdot,y) - v_n(\cdot,y_n^+ - y_n^-)\|_2 \leq 1$ for any $y > y_n^+ - y_n^-$ and (iii) follows with $\bar{C} = \frac{2\mu}{\mu-2}\beta + r_0 + \frac{m_b+1}{\sqrt{2(b^*-b)}} + 1$.
To prove (iv) we use (iii) and (2.9). By (2.9) we know that there exists $C > 0$ such that

$$V(v_n(\cdot,y)) \geq \tfrac{1}{2}\|\nabla v_n(\cdot,y)\|_2^2 \left(1 - C\frac{\|v_n(\cdot,y)\|_2^{(p+1)\theta}}{\|\nabla v_n(\cdot,y)\|_2^{2-(p+1)(1-\theta)}}\right) + \tfrac{1}{4}\|v_n(\cdot,y)\|_2^2 \quad \forall y \in \mathbb{R}.$$

We set

$$\mathcal{A}_n = \{y \in \mathbb{R} \,|\, \|\nabla v_n(\cdot,y)\|_2^{2-(p+1)(1-\theta)} \geq 2C\|v_n(\cdot,y)\|_2^{(p+1)\theta}\}.$$

By (2.9), $V(v_n(\cdot,y)) \geq \frac{1}{4}\|v_n(\cdot,y)\|^2$ for every $y \in \mathcal{A}_n$ while $\|\nabla v_n(\cdot,y)\|_2^{2-(p+1)(1-\theta)} < 2C\|v_n(\cdot,y)\|_2^{(p+1)\theta}$ for any $y \in \mathbb{R}\setminus\mathcal{A}_n$. By (iii) we know that $\|v_n(\cdot,y)\|_2 \leq \bar{C}$ for all $y \in \mathbb{R}$ and so $\|\nabla v_n(\cdot,y)\|_2^2 < \tilde{C} := 2C\bar{C}^{(p+1)\theta}$ for any $y \in \mathbb{R}\setminus\mathcal{A}_n$. Given $(y_1,y_2) \subset \mathbb{R}$ we have

$$
\begin{aligned}
\|v_n\|^2_{H^1(S_{((y_1,y_2))})} &= \int_{y_1}^{y_2} \|\partial_y v_n(\cdot,y)\|_2^2 + \|\nabla v_n(\cdot,y)\|_2^2 + \|v_n(\cdot,y)\|_2^2\,dy \\
&\leq 2\varphi(v_n) + \int_{y_1}^{y_2} \|\nabla v_n(\cdot,y)\|_2^2\,dy + \bar{C}(y_2-y_1) \\
&\leq 2\varphi(v_n) + \int_{(y_1,y_2)\cap\mathcal{A}_n} \|\nabla v_n(\cdot,y)\|_2^2\,dy + (\bar{C}+\tilde{C})(y_2-y_1) \\
&\leq 2\varphi(v_n) + 4\int_{(y_1,y_2)\cap\mathcal{A}_n} V(v_n(\cdot,y)) - b\,dy + (\bar{C}+\tilde{C}+4b)(y_2-y_1) \\
&\leq 6\varphi(v_n) + (\bar{C}+\tilde{C}+4b)(y_2-y_1) \\
&\leq \hat{C}^2 = 6(m_0+1) + (\bar{C}+\tilde{C}+4c)(y_2-y_1)
\end{aligned}
$$

and (iv) follows. □

By (iv) of Lemma 3.12 we have that the minimizing sequence (v_n) weakly converges in $H^1(S_L)$ for any $L > 0$ to a function $\bar{v} \in \mathcal{H}$. Even if we do not know a priori that $\bar{v} \in \mathcal{X}_b$, thanks to Lemma 3.3, Lemma 2.5 and the semicontinuity of the distance function, the function $\bar{v}$ enjoyes the following properties

Corollary 3.1 *For any $b \in [0,c)$ there exists $\bar{v} \in \mathcal{H}$ such that*

i) given any interval $I \subset \mathbb{R}$ such that $V(\bar{v}(\cdot,y)) \geq b$ for a.e. $y \in I$ we have $\varphi_I(\bar{v}) \leq m_b$,

ii) $\mathrm{dist}(\bar{v}(\cdot,y),\mathcal{V}_-^\beta) \leq r_0$ *for any* $y \leq 0$,

iii) $\mathrm{dist}(\bar{v}(\cdot,y),\mathcal{V}_+^\beta) \leq r_0$ *for any* $y \geq L_0$,

iv) $\|\bar{v}(\cdot,y)\|_2 \leq \bar{C}$ *for any* $y \in \mathbb{R}$,

v) for every $(y_1,y_2) \subset \mathbb{R}$, $\|\bar{v}\|_{H^1(S_{(y_1,y_2)})} \leq \hat{C}$,

where L_0, $\bar{C}$ and $\hat{C}$ are given by Lemma 3.12.

We define $\bar{\sigma}$ and $\bar{\tau}$ as follows:

$$
\begin{aligned}
\bar{\sigma} &= \sup\{y \in \mathbb{R}\,/\,\mathrm{dist}(\bar{v}(\cdot,y),\mathcal{V}_-^b) \leq r_0 \text{ and } V(\bar{v}(\cdot,y)) \leq b\}, \\
\bar{\tau} &= \inf\{y > \bar{\sigma}\,/\,V(\bar{v}(\cdot,y)) \leq b\},
\end{aligned}
$$

with the agreement that $\bar{\sigma} = -\infty$ whenever $V(\bar{v}(\cdot,y)) > b$ for every $y \in \mathbb{R}$ such that $\mathrm{dist}(\bar{v}(\cdot,y),\mathcal{V}_-^b) \leq r_0$ and that $\bar{\tau} = +\infty$ whenever $V(\bar{v}(\cdot,y)) > b$ for every $y > \bar{\sigma}$.

Remark 3.6 *Properties of $\bar{\sigma}$, $\bar{\tau}$:*

i) $\bar{\sigma} \in [-\infty, L_0]$ and $\bar{\tau} \in [0,+\infty]$.

By Corollary 3.1-(iii), if $y \geq L_0$ then $\mathrm{dist}(\bar{v}(\cdot,y),\mathcal{V}_+^\beta) \leq r_0$. Hence $\mathrm{dist}(\bar{v}(\cdot,y),\mathcal{V}_-^b) > 4r_0$ for $y \geq L_0$ and $\bar{\sigma} \leq L_0$ follows. Moreover, by Corollary 3.1-(ii), there results $\mathrm{dist}(\bar{v}(\cdot,y),\mathcal{V}_-^\beta) \leq r_0$ if $y \leq 0$. Then, by the definition of $\bar{\sigma}$, we have that if $\bar{\sigma} < 0$ then $V(\bar{v}(\cdot,y)) > b$ for any $y \in (\bar{\sigma},0]$ and so $\bar{\tau} \geq 0$ follows.

ii) If $\bar\sigma \in \mathbb{R}$ then $\bar v(\cdot,\bar\sigma) \in \mathcal{V}^b_-$.

Indeed, by definition, there exists a sequence $y_n \in (-\infty,\bar\sigma]$ such that $y_n \to \bar\sigma$ as $n \to +\infty$, $V(\bar v(\cdot,y_n)) \le b$ and $\mathrm{dist}(\bar v(\cdot,y_n),\mathcal{V}^b_-) \le r_0$ for any $n \in \mathbb{N}$. Then $\bar v(\cdot,y_n) \in \mathcal{V}^b_-$ for any $n \in \mathbb{N}$ and since, $\bar v(\cdot,y_n) \to \bar v(\cdot,\bar\sigma)$ in $L^2(\mathbb{R}^N)$, by Remark 2.5 we conclude that $\bar v(\cdot,\bar\sigma) \in \mathcal{V}^b_-$.

iii) $\bar\sigma < \bar\tau$.

It is sufficient to prove that if $\bar\sigma \in \mathbb{R}$ then, there exists $\delta > 0$ such that $V(\bar v(\cdot,y)) > b$ for any $y \in (\bar\sigma,\bar\sigma+\delta)$. Assume by contradiction that there exists a sequence $(y_n) \subset (\bar\sigma,+\infty)$ such that $V(\bar v(\cdot,y_n)) \le b$ for any $n \in \mathbb{N}$ and $y_n \to \bar\sigma$. Then, by definition of $\bar\sigma$ we have $\mathrm{dist}(\bar v(\cdot,y_n),\mathcal{V}^b_-) > r_0$ for any $n \in \mathbb{N}$ and so $\bar v(\cdot,y_n) \in \mathcal{V}^b_+$. Hence, since $v(\cdot,y_n) \to v(\cdot,\bar\sigma)$ in L^2, by Remark 2.6, we obtain $\bar v(\cdot,\bar\sigma) \in \mathcal{V}^b_+$ while, by (ii) we know that $\bar v(\cdot,\bar\sigma) \in \mathcal{V}^b_-$.

iv) If $\bar\tau \in \mathbb{R}$ then $\bar v(\cdot,\bar\tau) \in \mathcal{V}^b_+$.

Indeed, by definition, there exists a sequence $y_n \in [\bar\tau,+\infty)$ such that $y_n \to \bar\tau$ as $n \to +\infty$, $V(\bar v(\cdot,y_n)) \le b$. By definition of $\bar\sigma$, since $y_n > \bar\sigma$, we have $\mathrm{dist}(\bar v(\cdot,y_n),\mathcal{V}^b_-) > r_0$ for any $n \in \mathbb{N}$. Then $\bar v(\cdot,y_n) \in \mathcal{V}^b_+$ for any $n \in \mathbb{N}$ and since, $\bar v(\cdot,y_n) \to \bar v(\cdot,\bar\tau)$ in $L^2(\mathbb{R})$, we conclude by Remark 2.6 that $\bar v(\cdot,\bar\tau) \in \mathcal{V}^b_+$.

v) If $[y_1,y_2] \subset (\bar\sigma,\bar\tau)$ then $\inf_{y\in[y_1,y_2]} V(\bar v(\cdot,y)) > b$. Moreover $\varphi_{(\bar\sigma,\bar\tau)}(\bar v) \le m_b$.

It follows by the definition of $\bar\sigma$ and $\bar\tau$ that $V(\bar v(\cdot,y)) > b$ for any $y \in (\bar\sigma,\bar\tau)$. Then, by Lemma 3.1 we have $\inf_{y\in[y_1,y_2]} V(\bar v(\cdot,y)) = \min_{y\in[y_1,y_2]} V(\bar v(\cdot,y)) > b$ whenever $[y_1,y_2] \subset (\bar\sigma,\bar\tau)$. By Corollary 3.1–$(i)$ we furthermore derive that $\varphi_{(\bar\sigma,\bar\tau)}(\bar v) \le m_b$.

vi) If $\bar\sigma = -\infty$ then $\liminf_{y\to-\infty} V(\bar v(\cdot,y)) - b = \liminf_{y\to-\infty} \mathrm{dist}(\bar v(\cdot,y),\mathcal{V}^b_-) = 0$.

By Corollary 3.1–(ii) we have $\mathrm{dist}(\bar v(\cdot,y),\mathcal{V}^\beta_-) \le r_0$ for every $y \le 0$. Since $\bar\sigma = -\infty$ and $\varphi_{(-\infty,\bar\tau)}(\bar v) \le m_b$ we derive that there exists a sequence $y_n \to -\infty$ such that $V(\bar v(\cdot,y_n)) \to b$, $\bar v(\cdot,y_n) \in \mathcal{V}^\beta_-$ and $\mathrm{dist}(\bar v(\cdot,y_n),\mathcal{V}^b_+) \ge 4r_0$.
If $b = 0$, by Remark 2.5, we obtain $\bar v(\cdot,y_n) \to 0$ and (vi) follows. If $b > 0$, arguing as in the proof of Lemma 3.6, for any $n \in \mathbb{N}$, since $V(\bar v(\cdot,y_n)) > b$, there exists of a unique $s_n \in (0,1]$ such that $V(s_n\bar v(\cdot,y_n)) = b$, $s_n\bar v(\cdot,y_n)) \in \mathcal{V}^b_-$ with $1 - s_n \le (V(\bar v(\cdot,y_n)) - b)/\nu^-(b) \to 0$. Since by Remark 2.5 $\|\bar v(\cdot,y_n)||_2$ is bounded, $\mathrm{dist}(\bar v(\cdot,y_n),\mathcal{V}^b_-) \le (1 - s_n)\|\bar v(\cdot,y_n)\|_2 \to 0$ and (vi) follows.

vii) If $\bar\tau = +\infty$ then $\liminf_{y\to+\infty} V(\bar v(\cdot,y)) - b = \liminf_{y\to+\infty} \mathrm{dist}(\bar v(\cdot,y),\mathcal{V}^b_+) = 0$.

The proof is analogous of the one of (vi).

Thank to properties (ii)–(iv), (vi)–(vii) we recognize that the function $\bar v$ satisfies the assumption of Lemma 3.4 on the interval $(\bar\sigma,\bar\tau)$ which allows us to derive the following properties of $\bar v$.

Lemma 3.13 *There result*

i) $\varphi_{(\bar\sigma,\bar\tau)}(\bar v) = m_b$ *and* $\liminf_{y\to\bar\tau^-} V(\bar v(\cdot,y)) = \liminf_{y\to\bar\sigma^+} V(\bar v(\cdot,y)) = b$,

ii) $\bar\tau \in \mathbb{R}$ *for any* $b \in [0,c)$ *and* $\bar\sigma \in \mathbb{R}$ *for any* $b \in (0,c)$,

iii) for every $h \in C^\infty_0(\mathbb{R}^N \times (\bar\sigma,\bar\tau))$, *with* $\mathrm{supp}\, h \subset \mathbb{R}^N \times [y_1,y_2] \subset \mathbb{R}^N \times (\bar\sigma,\bar\tau)$, *there exists* $\bar t > 0$ *such that*

$$\varphi_{(\bar\sigma,\bar\tau)}(\bar v + th) \ge \varphi_{(\bar\sigma,\bar\tau)}(\bar v), \quad \forall\, t \in (0,\bar t). \tag{3.25}$$

Then $\bar v \in C^2(\mathbb{R}^N \times (\bar\sigma,\bar\tau))$ verifies $-\Delta u + u - f(u) = 0$ on $\mathbb{R}^N \times (\bar\sigma,\bar\tau)$ and for any $[y_1,y_2] \subset (\bar\sigma,\bar\tau)$ there results $\bar v \in H^2(\mathbb{R}^N \times (y_1,y_2))$,

iv) $E_y(\bar v(\cdot,y)) = \frac{1}{2}\|\partial_y \bar v(\cdot,y)\|_2^2 - V(\bar v(\cdot,y)) = -b$ *for every* $y \in (\bar\sigma,\bar\tau)$*,*

v) $\liminf_{y\to\bar\tau^-} \|\partial_y \bar v(\cdot,y)\|_2 = \liminf_{y\to\bar\sigma^+} \|\partial_y \bar v(\cdot,y)\|_2 = 0.$

Proof. (i) By Lemma 3.4 we already know that $\varphi_{(\bar\sigma,\bar\tau)}(\bar v) \geq m_b$ and by (v) of Remark 3.6 we conclude that $\varphi_{(\bar\sigma,\bar\tau)}(\bar v) = m_b$. Hence, using Lemma 3.4 again, we conclude that $\liminf_{y\to\bar\tau^-} V(\bar v(\cdot,y)) = \liminf_{y\to\bar\sigma^+} V(\bar v(\cdot,y)) = b$.

(ii) Assume by contradiction that $\bar\tau = +\infty$. By (vii) of Remark 3.6 there exists $y_0 > L_0$ such that $u_0 := \bar v(\cdot,y_0) \in \mathcal{V}_+^{\beta_+} \setminus \mathcal{V}_+^b$ and $\bar v(\cdot,\cdot + y_0) \in \mathcal{X}^+_{b,u_0}$. To obtain a contradiction we show that

$$\forall y \geq y_0 \text{ such that } V(\bar v(\cdot,y)) \leq \beta_+ \text{ we have } \varphi_{(y,+\infty)}(\bar v) \leq C_+(b)(V(\bar v(\cdot,y)) - b)^{3/2}. \quad (3.26)$$

By (3.26), using Lemma 3.7, we derive that there exists $\bar y \in (y_0, y_0+1)$ such that $V(\bar v(\cdot,\bar y)) = b$ which contradicts that $\bar\tau = +\infty$.
If (3.26) does not hold, by Lemma 3.6, there exists $\tilde y \geq y_0$ with $\bar v(\cdot,\tilde y) \in \mathcal{V}_+^{\beta_+}$ and $\varphi_{(\tilde y,+\infty)}(\bar v) > \varphi_{(\tilde y,+\infty)}(w^+_{\bar v(\cdot,\tilde y)})$. Then, defining

$$\tilde v(\cdot,y) = \begin{cases} \bar v(\cdot,y) & y \leq \tilde y \\ w^+_{\bar v(\cdot,\tilde y)}(\cdot,\cdot - \tilde y) & y > \tilde y \end{cases}$$

we obtain $\varphi_{(\bar\sigma,+\infty)}(\tilde v) < \varphi_{(\bar\sigma,+\infty)}(\bar v) = m_b$. On the other hand, defining $\tilde\tau = \sup\{y > \bar\sigma \,|\, V(\tilde v(\cdot,y)) > b\}$, we recognize that $\tilde\tau$ satisfies the assumption of Lemma 3.4 on the interval $(\bar\sigma,\tilde\tau)$ and we get the contradiction $m_b \leq \varphi_{(\bar\sigma,\tilde\tau)}(\tilde v) \leq \varphi_{(\bar\sigma,+\infty)}(\tilde v) < m_b$.
To prove that $\bar\sigma \in \mathbb{R}$ when $b > 0$ we can argue analogously using Lemmas 3.9 and 3.10.

(iii) Let us consider $h \in C_0^\infty(\mathbb{R}^N \times (\bar\sigma,\bar\tau))$ with $\operatorname{supp} h \subset \mathbb{R}^N \times [y_1,y_2] \subset \mathbb{R}^N \times (\bar\sigma,\bar\tau)$. By (v) of Remark 3.6 we know that there exists $\mu > 0$ such that $V(\bar v(\cdot,y)) \geq b + \mu$ for any $y \in [y_1,y_2]$. Let us consider $(\bar v + th)^*$ the symmetric decreasing rearrangement of the function $v + th$ with respect to the variable x, i.e. the unique function with radial symmetry with respect to the variable $x \in \mathbb{R}^N$ such that

$$|\{x \in \mathbb{R}^N \,|\, (\bar v + th)^*(\cdot,y) > r\}| = |\{x \in \mathbb{R}^N \,|\, |(\bar v + th)(\cdot,y)| > r\}| \text{ for every } r > 0 \text{ and a.e. } y \in \mathbb{R}$$

and $(\bar v + th)^*(x_1,y) \geq (\bar v + th)^*(x_2,y)$ whenever $|x_1| \leq |x_2|$, for a.e. $y \in \mathbb{R}$. One recognizes (use e.g. [19], (12)-(14), and [22], (3) pg. 73) that $\|\nabla(\bar v + th)^*\|_{L^2(\mathbb{R}^N\times(y_1,y_2))} \leq \|\nabla(\bar v + th)\|_{L^2(\mathbb{R}^N\times(y_1,y_2))}$ and $\int_{\mathbb{R}^N\times(y_1,y_2)} \frac{1}{2}|(\bar v + th)^*|^2 + F((\bar v + th)^*)\,dxdy = \int_{\mathbb{R}^N\times(y_1,y_2)} \frac{1}{2}|v + th|^2 + F(|v + th|)\,dxdy = \int_{\mathbb{R}^N\times(y_1,y_2)} \frac{1}{2}|v + th|^2 + F(\bar v + th)\,dxdy$. Therefore we have

$$\begin{aligned}\int_{\mathbb{R}^N\times[y_1,y_2]} &\tfrac{1}{2}|\nabla(\bar v + th)^*|^2 + \tfrac{1}{2}|(\bar v + th)^*|^2 - F((\bar v + th)^*)\,dxdy \\ &\leq \int_{\mathbb{R}^N\times[y_1,y_2]} \tfrac{1}{2}|\nabla(\bar v + th)|^2 + \tfrac{1}{2}|\bar v + th|^2 - F(\bar v + th)\,dxdy \end{aligned} \quad (3.27)$$

We now claim that

$$\exists\,\bar t > 0 \text{ such that } V((\bar v + th)^*(\cdot,y)) > b + \mu/2 \text{ for any } t \in [0,\bar t] \text{ and } y \in [y_1,y_2]. \quad (3.28)$$

Arguing by contradiction, if (3.28) does not hold, there exists a sequence $t_n \in (0,1)$ and a sequence $y_n \in [y_1, y_2]$ such that $t_n \to 0$, $y_n \to y_0 \in [y_1, y_2]$ and $V((\bar{v}+t_n h)^*(\cdot, y_n)) \le b+\mu/2$. By Corollary 3.1-$(iv)$, since h has compact support, We have that there exists $C > 0$ such that $\|(\bar{v}+t_n h)^*(\cdot, y_n)\|_2 = \|(\bar{v}+t_n h)(\cdot, y_n)\|_2 \le \|\bar{v}(\cdot, y_n)\|_2 + \|h(\cdot, y_n)\|_2 \le C$ for any $n \in \mathbb{N}$. Since $V((\bar{v}+t_n h)^*(\cdot, y_n)) \le b+\mu/2$, by Lemma 2.8 there exists a constant $R > 0$ such that $\|\nabla(\bar{v}+t_n h)^*(\cdot, y_n)\|_2 \le R$ for any $n \in \mathbb{N}$. Then the sequence $\{(\bar{v}+t_n h)^*(\cdot, y_n)\}$ is bounded in $H^1(\mathbb{R}^N)$. Since the rearrangement is contractive in $L^2(\mathbb{R}^N)$ we have $\|(\bar{v}+t_n h)^*(\cdot, y_n) - \bar{v}(\cdot, y_0)\|_2 \le \|(\bar{v}+t_n h)(\cdot, y_n) - \bar{v}(\cdot, y_0)\|_2 \to 0$ and so $(\bar{v}+t_n h)^*(\cdot, y_n) \to \bar{v}(\cdot, y_0)$ weakly in $H^1(\mathbb{R}^N)$. By Lemma 2.4 we then obtain the contradiction $b+\mu/2 \ge \liminf_{n\to+\infty} V((\bar{v}+t_n h)^*(\cdot, y_n)) \ge V(\bar{v}(\cdot, y_0)) \ge b+\mu$ which proves (3.28).
Since $\bar{v}(\cdot, y) \in \mathcal{X}$ for a.e. $y \in \mathbb{R}$ we have $\bar{v} = \bar{v}^*$ and $(\bar{v}+th)^* = v$ for $x \in \mathbb{R}^N$ and $y \in \mathbb{R} \setminus [y_1, y_2]$. By (3.28) we then recognize that $(\bar{v}+th)^*$ satisfies the assumptions of Lemma 3.4 on the interval $(\bar{\sigma}, \bar{\tau})$ for any $t \in [0, \bar{t}]$. Then $\varphi_{(\bar{\sigma},\bar{\tau})}((\bar{v}+th)^*) \ge m_b = \varphi_{(\bar{\sigma},\bar{\tau})}(\bar{v})$ for any $t \in [0, \bar{t}]$ and (3.25) follows by (3.27). Finally, by (3.25) we have

$$\int_{\mathbb{R}^N\times(\bar{\sigma},\bar{\tau})} \tfrac{1}{2}|\nabla(\bar{v}+th)|^2 + \tfrac{1}{2}|\bar{v}+th|^2 - F(\bar{v}+th) - \tfrac{1}{2}|\nabla\bar{v}|^2 - \tfrac{1}{2}|\bar{v}|^2 + F(\bar{v})\,dxdy \ge 0 \quad \forall t \in (0, \bar{t}).$$

Since h is arbitrary we derive that $\int_{\mathbb{R}^N\times(\bar{\sigma},\bar{\tau})} \nabla\bar{v}\nabla h + \bar{v}\cdot h - f(\bar{v})h\,dx\,dy = 0$ for every $h \in C_0^\infty(\mathbb{R}^N \times (\bar{\sigma}, \bar{\tau}))$, and so that $\bar{v}$ is a weak solution of (E) on $\mathbb{R}^N \times (\bar{\sigma}, \bar{\tau})$. Then (iii) follows by (v) of Corollary 3.1 and standard regularity arguments.

(iv) Fixed $\xi \in (\bar{\sigma}, \bar{\tau})$ and $s > 0$ we define

$$\bar{v}_s(\cdot, y) = \begin{cases} \bar{v}(\cdot, y+\xi) & y \le 0, \\ \bar{v}(\cdot, \frac{y}{s}+\xi) & 0 < y. \end{cases}$$

and we note that $\bar{v}_s$ verifies the assumption of Lemma 3.4 on the interval $(\bar{\sigma}-\xi, s(\bar{\tau}-\xi))$. Then

$$\varphi_{(\bar{\sigma}-\xi,s(\bar{\tau}-\xi))}(\bar{v}_s) \ge m_p = \varphi_{(\bar{\sigma}-\xi,\bar{\tau}-\xi)}(\bar{v}(\cdot, \cdot+\xi))$$

and so we have that for any $s > 0$ there results

$$\begin{aligned}
0 &\le \varphi_{(\bar{\sigma}-\xi,s(\bar{\tau}-\xi))}(\bar{v}_s) - \varphi_{(\bar{\sigma}-\xi,\bar{\tau}-\xi)}(\bar{v}(\cdot, \cdot+\xi)) \\
&= \int_0^{s(\bar{\tau}-\xi)} \tfrac{1}{2}\|\partial_y \bar{v}_s(\cdot, y)\|^2 + (V(\bar{v}_s(\cdot, y)) - b)\,dy - \int_\xi^{\bar{\tau}} \tfrac{1}{2}\|\partial_y \bar{v}(\cdot, y)\|^2 + (V(\bar{v}(\cdot, y)) - b)\,dy \\
&= \int_0^{s(\bar{\tau}-\xi)} \tfrac{1}{2s^2}\|\partial_y \bar{v}(\cdot, \tfrac{y}{s}+\xi)\|^2 + (V(\bar{v}(\cdot, \tfrac{y}{s}+\xi)) - b)\,dy - \varphi_{(\xi,\bar{\tau})}(u) \\
&= \tfrac{1}{s}\int_\xi^{\bar{\tau}} \tfrac{1}{2}\|\partial_y \bar{v}(\cdot, y)\|^2\,dy + s\int_\xi^{\bar{\tau}} V(\bar{v}(\cdot, y)) - b\,dy - \varphi_{(\xi,\bar{\tau})}(\bar{v}) \\
&= (\tfrac{1}{s}-1)\int_\xi^{\bar{\tau}} \tfrac{1}{2}\|\partial_y \bar{v}(\cdot, y)\|^2\,dy + (s-1)\int_\xi^{\bar{\tau}} V(\bar{v}(\cdot, y)) - b\,dy.
\end{aligned}$$

This means that, setting $A = \int_\xi^{\bar{\tau}} \frac{1}{2}\|\partial_y \bar{v}(\cdot, y)\|^2\,dy$ and $B = \int_\xi^{\bar{\tau}} V(\bar{v}(\cdot, y)) - b\,dy$, the real function $s \mapsto \psi(s) = A(\frac{1}{s}-1) + B(s-1)$ is non negative on $(0, +\infty)$ and then that $0 \le \min\psi(s) = \psi(\sqrt{\frac{A}{B}}) = -(\sqrt{A}-\sqrt{B})^2$, that implies $A = B$, i.e.,

$$\int_\xi^{\bar{\tau}} V(\bar{v}(\cdot, y)) - b\,dy = \int_\xi^{\bar{\tau}} \tfrac{1}{2}\|\partial_y \bar{v}(\cdot, y)\|_2^2\,dy \text{ for any } \xi \in (\bar{\sigma}, \bar{\tau}). \tag{3.29}$$

Since, by (iii), $\bar v \in H^2(\mathbb{R}^N \times (y_1, y_2))$ whenever $[y_1, y_2] \subset (\bar\sigma, \bar\tau)$, we derive that the function $y \to \frac{1}{2}\|\partial_y \bar v(\cdot, y)\|_2^2 - V(\bar v(\cdot, y))$ is continuous and (iv) follows by (3.29).

(v) It follows by (i) and (iv). $\square$

3.4 The case $b > 0$. The periodic solutions

Consider the case $b \in (0, c)$. By (ii) of Lemma 3.13 we have $\bar\sigma, \bar\tau \in \mathbb{R}$. In this case, by reflection and periodic continuation, starting from $\bar v$, we can construct a solution to (E) on all $\mathbb{R}^{N+1}$ periodic in the variable y. Precisely let

$$v(x,y) = \begin{cases} \bar v(x, y + \bar\sigma) & \text{if } x \in \mathbb{R}^N \text{ and } y \in [0, \bar\tau - \bar\sigma) \\ \bar v(x, \bar\tau + (\bar\tau - \bar\sigma - y)) & \text{if } x \in \mathbb{R}^N \text{ and } y \in [\bar\tau - \bar\sigma, 2(\bar\tau - \bar\sigma)] \end{cases}$$

and $v(x,y) = v(x, y + 2k(\bar\tau - \bar\sigma))$ for all $(x,y) \in \mathbb{R}^{N+1}$, $k \in \mathbb{Z}$.

Remark 3.7 Let $T = \bar\tau - \bar\sigma$.

- *i)* The function $y \in \mathbb{R} \mapsto v(\cdot, y) \in L^2(\mathbb{R}^N)$ is continuous and periodic with period $2T$. Moreover by (ii) and (iv) of Remark 3.6, $v(\cdot, 0) \in \mathcal{V}_-^b$ and $v(\cdot, T) \in \mathcal{V}_+^b$. Finally, by definition, $v(\cdot, -y) = v(\cdot, y)$ and $v(\cdot, y+T) = v(\cdot, T-y)$ for any $y \in \mathbb{R}$.
- *ii)* $v \in \mathcal{H}$ and, by (v) of Remark 3.6, $V(v(\cdot, y)) > b$ for any $y \in \mathbb{R} \setminus \{kT \,/\, k \in \mathbb{Z}\}$.
- *iii)* By (v) of Lemma 3.13, for any $k \in \mathbb{Z}$ we have $\liminf_{y \to kT^{\pm}} \|\partial_y v(\cdot, y)\|_2 = 0$.
- *iv)* By (iii) of Lemma 3.13 , $v \in C^2(\mathbb{R}^N \times (0,T))$ satisfies $-\Delta v(x,y) + v(x,y) - f(v(x,y)) = 0$ for $(x,y) \in \mathbb{R}^N \times (0,T)$.

We have

Lemma 3.14 *$v \in \mathcal{C}^2(\mathbb{R}^{N+1})$ is a solution of (E) on $\mathbb{R}^{N+1}$. Moreover, $E_v(y) = \frac{1}{2}\|\partial_y v(\cdot, y)\|_2^2 - V(v(\cdot, y)) = -b$ for all $y \in \mathbb{R}$ and $\partial_y v(\cdot, 0) = \partial_y v(\cdot, T) = 0$. Finally $v > 0$ on $\mathbb{R}^{N+1}$.*

Proof. First, let us prove that v is a classical solution to (E). To this aim, we first note that by Remark 3.7 (iii), there exist four sequences $(\varepsilon_n^{\pm}), (\eta_n^{\pm})$, such that $\varepsilon_n^- < 0 < \varepsilon_n^+$, $\eta_n^- < 0 < \eta_n^+$ for any $n \in \mathbb{N}$, $\varepsilon_n^{\pm}$, $\eta_n^{\pm} \to 0$ and

$$\lim_{n \to +\infty} \|\partial_y v(\cdot, \varepsilon_n^{\pm})\|_2 = \lim_{n \to +\infty} \|\partial_y v(\cdot, T + \eta_n^{\pm})\|_2 = 0. \tag{3.30}$$

Fixed any $\psi \in C_0^\infty(\mathbb{R}^{N+1})$, by Remark 3.7 (i)-(iv) we obtain that for any $k \in \mathbb{Z}$ and n sufficiently large we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} \int_{2kT+\varepsilon_n^+}^{(2k+1)T+\eta_n^-} -\Delta v\,\psi + v\psi - f(v)\psi \,dy\,dx \\ &= \int_{\mathbb{R}^N} \int_{2kT+\varepsilon_n^+}^{(2k+1)T+\eta_n^-} \nabla v \nabla\psi + v\psi - f(v)\psi \,dy\,dx + \int_{\mathbb{R}^N} \partial_y v(x, 2kT + \varepsilon_n^+)\psi(x, 2kT + \varepsilon_n^+)\,dx \\ &\quad - \int_{\mathbb{R}^N} \partial_y v(x, (2k+1)T + \eta_n^-)\psi(x, (2k+1)T + \eta_n^-)\,dx \end{aligned}$$

and

$$
\begin{aligned}
0 &= \int_{\mathbb{R}^N} \int_{(2k-1)T+\eta_n^+}^{2kT+\varepsilon_n^-} -\Delta v\, \psi + +v\psi - f(v)\psi \, dy \, dx \\
&= \int_{\mathbb{R}^N} \int_{(2k-1)T+\eta_n^+}^{2kT+\varepsilon_n^-} \nabla v \nabla \psi + +v\psi - f(v)\psi \, dy \, dx - \int_{\mathbb{R}^N} \partial_y v(x, 2kT + \varepsilon_n^-)\psi(x, 2kT + \varepsilon_n^-) \, dx \\
&\quad + \int_{\mathbb{R}^N} \partial_y v(x, (2k-1)T + \eta_n^+)\psi(x, (2k-1)T + \eta_n^+) \, dx.
\end{aligned}
$$

By (3.30), in the limit for $n \to +\infty$, we obtain that for any $k \in \mathbb{Z}$ we have

$$
0 = \int_{\mathbb{R}^N} \int_{(2k-1)T}^{2kT} \nabla v \nabla \psi + v\psi - f(v)\psi \, dy \, dx = \int_{\mathbb{R}^N} \int_{2kT}^{(2k+1)T} \nabla v \nabla \psi + v\psi - f(v)\psi \, dy \, dx.
$$

Then, v satisfies

$$
\int_{\mathbb{R}^{N+1}} \nabla v \nabla \psi + +v\psi - f(v)\psi \, dx \, dy = 0, \qquad \forall \psi \in C_0^\infty(\mathbb{R}^{N+1})
$$

and so v is a *classical* solution to (E) on $\mathbb{R}^{N+1}$ which is periodic of period $2T$ in the variable y. Since by (v) of Corollary 3.1 we have $\|\bar{v}(\cdot, y)\|_{H^1(S_{(0,T)})} \le \hat{C}$ depending only on T, by definition of v and using (E) we recover that $v \in H^2(\mathbb{R}^N \times (y_1, y_2))$ for any bounded interval $(y_1, y_2) \subset \mathbb{R}$ and $\|v\|_{H^2(S_{(y_1,y_2)})} \le C$ with C depending only on $y_2 - y_1$. This implies in particular that the functions $y \in \mathbb{R} \to \partial_y v(\cdot, y) \in L^2(\mathbb{R}^N)$ and $y \in \mathbb{R} \to v(\cdot, y) \in H^1(\mathbb{R}^N)$ are uniformly continuous. Then $\lim_{y \to 0^+} V(v(\cdot, y)) - b = \liminf_{y \to 0^+} \|\partial_y v(\cdot, y)\|_2 = 0$ and analogously $\lim_{y \to T^-} V(v(\cdot, y)) - b = \lim_{y \to T^-} \|\partial_y v(\cdot, y)\|_2 = 0$. By continuity we derive that $\partial_y v(\cdot, 0) = \partial_y v(\cdot, T) = 0$. By (v) of Lemma 3.13 and the definition of v it then follows that $\frac{1}{2}\|\partial_y v(\cdot, y)\|^2 - V(v(\cdot, y)) = -b$ for any $y \in \mathbb{R}$.
To complete the proof we have to show that $v > 0$ on $\mathbb{R}^{N+1}$. We know that $v \ne 0$ and since $v \in \mathcal{H}$ we have $v \ge 0$ on $\mathbb{R}^{N+1}$. Since v solves (E) we have $-\Delta v + v = f(v) \ge 0$ on $\mathbb{R}^{N+1}$ and $v > 0$ on $\mathbb{R}^{N+1}$ follows from the strong maximum principle. $\square$

Lemma 3.15 *We have $\partial_y v > 0$ on $\mathbb{R}^N \times (0, T)$ and $\partial_{x_i} v < 0$ on $\{(x, y) \in \mathbb{R}^{N+1} \,|\, x_i > 0\}$ for every $i = 1, ..., N$.*

Proof. To prove that $\partial_y v > 0$ on $\mathbb{R}^N \times (0, T)$ we first note that since $\partial_y v(\cdot, 0) = \partial_y v(\cdot, T) = 0$ then $\partial_y v \in H_0^1(\mathbb{R}^N \times (0, T))$ and solves the linear elliptic equation $-\Delta \partial_y v + \partial_y v - f'(v)\partial_y v = 0$ on $\mathbb{R}^N \times (0, T)$. Then $\partial_y v \in H_0^1(\mathbb{R}^N \times (0, T)) \cap H^2(\mathbb{R}^N \times (0, T))$ is an eigenfunction of the linear selfadjoint operator $\mathcal{L}_v : H_0^1(\mathbb{R}^N \times (0, T)) \cap H^2(\mathbb{R}^N \times (0, T)) \subset L^2(\mathbb{R}^N \times (0, T)) \to L^2(\mathbb{R}^N \times (0, T))$ defined by $\mathcal{L}_v h = -\Delta h + h - f'(v)h$ corresponding to the eigenvalue 0.
The minimality property of v proved in Lemma 3.13-(iii) implies $\langle \mathcal{L}_v h, h \rangle_2 \ge 0$ for any $h \in C_0^\infty(\mathbb{R}^N \times (0, T))$ and we deduce that 0 is the minimal eigenvalue of $\mathcal{L}_v$. Then $\partial_y v$ has constant sign on $\mathbb{R}^N \times (0, T)$. Assume by contradiction that $\partial_y v < 0$ on $\mathbb{R}^N \times (0, T)$. Since, by construction, v is even with respect to T, that implies that $v(x, T) \le v(x, y)$ for all $x \in \mathbb{R}^N$ and $0 < y < 2T$. We deduce that $\partial_{y,y}^2 v(x, T) \ge 0$ for all $x \in \mathbb{R}^N$ and so, multiplying (E) by v and recalling that $v > 0$ on $\mathbb{R}^{N+1}$ we deduce $-\Delta_x v(x, T)\, v(x, T) + v(x, T)^2 - f(v)v \ge 0$. Integrating with respect to x on $\mathbb{R}^N$ we obtain $V'(v(\cdot, T))v(\cdot, T) \ge 0$ contrary to the fact that $v(\cdot, T) \in \mathcal{V}_+^b$. This shows that $\partial_y v > 0$ on $\mathbb{R}^N \times (\bar{\sigma}, \bar{\tau})$. To prove that $\partial_{x_i} v < 0$ on $\{(x, y) \in \mathbb{R}^{N+1} \,|\, x_i > 0\}$ we note that since $v \in \mathcal{H} \cap C^2(\mathbb{R}^{N+1})$ we have $\partial_{|x|} v(x, y) \le 0$ for

all $y \in \mathbb{R}$ and $|x| \neq 0$. Then $\partial_{x_i} v \leq 0$ on $\{(x,y) \in \mathbb{R}^{N+1} \,|\, x_i > 0\}$. Since $\partial_{x_i} v \leq 0$ solves the linear elliptic equation $-\Delta \partial_{x_i} v + \partial_{x_i} v = f'(v)\partial_{x_i} v$ on $\{(x,y) \in \mathbb{R}^{N+1} \,|\, x_i > 0\}$ we deduce $-\Delta \partial_{x_i} v + \partial_{x_i} v \leq (f'(v))_+ \partial_{x_i} v \leq 0$ on $\{(x,y) \in \mathbb{R}^{N+1} \,|\, x_i > 0\}$ and since $\partial_{x_i} v \neq 0$, the strong maximum principle assures $\partial_{x_i} v < 0$ on $\{(x,y) \in \mathbb{R}^{N+1} \,|\, x_i > 0\}$. $\square$

3.5 The case $b = 0$. The homoclinic type mountain pass solution.

In the case $b = 0$ Lemma 3.13 establishes that $\bar{\tau} \in \mathbb{R}$ but does not give information about $\bar{\sigma}$. We prove here below that in fact $\bar{\sigma} = -\infty$.

Lemma 3.16 *If $b = 0$ then $\bar{\sigma} = -\infty$.*

Proof. Assume that $\bar{\sigma} \in \mathbb{R}$. Then, arguing as in the case $b > 0$, by reflection and periodic continuation, we construct a solution $v \in C^2(\mathbb{R}^{N+1})$ of (E) which is $2(\bar{\tau} - \bar{\sigma})$-periodic in the variable y with $v(\cdot, 0) \in \mathcal{V}^0_-$ and $\partial_y v(\cdot, 0) = 0$. Since $\mathcal{V}^0_- = \{0\}$ we have $v(x, 0) = 0$ and $\partial_y v(x, 0) = 0$ for any $x \in \mathbb{R}^N$. Defining $a(x,y) = 1 - f(v(x,y))/v(x,y)$ when $v(x,y) \neq 0$ and $a(x,y) = 1 - f'(0) = 1$ when $v(x,y) = 0$ we have that a is continuous on $\mathbb{R}^{N+1}$ and v solves $-\Delta v + a(x,y)v = 0$ on $\mathbb{R}^{N+1}$. Defining the function $\tilde{v}(\cdot, y) = v(\cdot, y)$ for $y \in (0, 2(\bar{\tau} - \bar{\sigma}))$, and $\tilde{v}(\cdot, y) = 0$ for $y \leq 0$ or $y \geq 2(\bar{\tau} - \bar{\sigma})$, since $v(x, 0) = \partial_y v(x, 0) = v(x, 2(\bar{\tau} - \bar{\sigma})) = \partial_y v(x, 2(\bar{\tau} - \bar{\sigma})) = 0$, we obtain that also $\tilde{v}$ satisfies $-\Delta v + a(x,y)v = 0$ on $\mathbb{R}^{N+1}$. But a local unique continuation theorem (see e.g. Theorem 5 in [23]) and a continuation argument imply that $\tilde{v} = 0$ on $\mathbb{R}^{N+1}$ while, by definition of $\bar{\sigma}$ and $\bar{\tau}$, $\tilde{v}(\cdot, y) = v(\cdot, y) = \bar{v}(\cdot, y + \bar{\sigma}) \neq 0$ for $y \in (0, \bar{\tau} - \bar{\sigma})$. $\square$

By Lemma 3.16 we can define the function

$$v(x,y) = \begin{cases} \bar{v}(x, y + \bar{\tau}) & \text{if } x \in \mathbb{R}^N \text{ and } y \in (-\infty, 0] \\ \bar{v}(x, \bar{\tau} - y) & \text{if } x \in \mathbb{R}^N \text{ and } y \in [0, +\infty) \end{cases}$$

and the argument of the proof of Lemma 3.14 shows that v is a classical solution to (E) in $\mathbb{R}^{N+1}$.

Remark 3.8 Again by (v) of Corollary 3.1 and using (E) we recover that $v \in H^2(\mathbb{R}^N \times (y_1, y_2))$ for any bounded interval $(y_1, y_2) \subset \mathbb{R}$ and $\|v\|_{H^2(S_{(y_1,y_2)})} \leq C$ with C depending only on $y_2 - y_1$. This implies in particular that the functions $y \in \mathbb{R} \to \partial_y v(\cdot, y) \in L^2(\mathbb{R}^N)$ and $y \in \mathbb{R} \to v(\cdot, y) \in H^1(\mathbb{R}^N)$ are uniformly continuous and so $\lim_{y \to -\infty} V(v(\cdot, y)) = \liminf_{y \to +\infty} V(v(\cdot, y)) = 0$, $\lim_{y \to 0^-} \|\partial_y v(\cdot, y)\|_2 = \liminf_{y \to 0^+} \|\partial_y v(\cdot, y)\|_2 = 0$, and $E_v(y) = \frac{1}{2}\|\partial_y v(\cdot, y)\|_2^2 - V(v(\cdot, y)) = 0$ for any $y \in \mathbb{R}$. Note finally that v is radially symmetric with respect to x, not increasing with respect to $|x|$, and, by construction, even in the variable y.

In the case $b = 0$ the functional $\varphi(u) = \int_{\mathbb{R}} \frac{1}{2}\|\partial_y u(\cdot, y)\|_2^2 + V(u(\cdot, u))\, dy$ can be written, by Remark 2.2, as $\varphi(u) = \frac{1}{2}\|u\|^2_{H^1(\mathbb{R}^{N+1})} - \int_{\mathbb{R}^{N+1}} F(u) dx dy = V_{N+1}(u)$ for all $u \in H^1(\mathbb{R}^{N+1})$ and in particular, denoting c_{N+1} the mountain pass level of φ in $H^1(\mathbb{R}^{N+1})$, Proposition 2.1 establishes that φ has a positive radially symmetric critical point w at the level c_{N+1}. We have

Lemma 3.17 *$v \in H^1(\mathbb{R}^{N+1})$ is a critical point of φ on $H^1(\mathbb{R}^{N+1})$ with $\varphi(v) = c_{N+1}$. Moreover $v \in \mathcal{C}^2(\mathbb{R}^N)$ is a positive solution of (E) on $\mathbb{R}^{N+1}$ such that $v(x,y) \to 0$ as $|(x,y)| \to +\infty$, and, up to translations, v is radially symmetric about the origin and $\partial_r v < 0$ for $r = |(x,y)| > 0$.*

Proof. By Remark 3.8 we have $\lim_{y\to-\infty} V(v(\cdot,y)) = 0$ and so there exists $y_0 \le -L_0 < 0$ (L_0 as in Corollary 3.1) such that $V(v(\cdot,y)) \le \beta$ for any $y \le y_0$. Since by Corollary 3.1-(ii) we know that $\mathrm{dist}(v(\cdot,y), \mathcal{V}_+^\beta) \ge 4r_0$ for $y \le -L_0$, we recognize that $v(\cdot,y) \in \mathcal{V}_-^\beta$ for any $y \le y_0$. Then $V'(v(\cdot,y))v(\cdot,y) \ge 0$ and by (2.2) we obtain that $V(v(\cdot,y)) \ge \frac{\mu-2}{2\mu}\|v(\cdot,y)\|^2$ for any $y \le y_0$. Since $m_0 = \varphi_{(-\infty,0)}(\bar v) \ge \int_{(-\infty,y_0)} V(v(\cdot,y))\,dy$, using Corollary 3.1-(iv), we then obtain $\|v\|^2_{H^1(\mathbb{R}^{N+1})} = 2\int_{-\infty}^0 \|v(\cdot,y)\|^2\,dy \le 2\int_{-\infty}^{y_0} \frac{2\mu}{\mu-2} V(v(\cdot,y))\,dy + 2\bar C|y_0| \le \frac{4\mu}{\mu-2} m_0 + 2\bar C|y_0|$, and $v \in H^1(\mathbb{R}^{N+1})$ follows. Since $v \in H^1(\mathbb{R}^{N+1})$ solves (E) on $\mathbb{R}^{N+1}$, we deduce $v(x,y) \to 0$ as $|(x,y)| \to +\infty$, it is a critical point of φ on $H^1(\mathbb{R}^{N+1})$ and, by Remark 2.1, $\varphi(v) = 2m_0 \ge c_{N+1}$.
We now show that $2m_0 \le c_{N+1}$ and the Lemma will follows as in the proof of Proposition 2.1.
As recalled above, φ admits on $H^1(\mathbb{R}^{N+1})$ a positive, radially symmetric (in $\mathbb{R}^{N+1}$) critical point w such that $\varphi(w) = c_{N+1}$ and $\partial_r w < 0$ on $\mathbb{R}^{N+1}\setminus\{0\}$ where $r = |(x,y)|$. In particular $w(x,y)$ is radially symmetric with respect to x and monotone decreasing with respect to $|x|$ for any $y \in \mathbb{R}$ and so $w \in \mathcal{H}$. By Lemma 3.2 we know that the *energy* function $E_w(y) = \frac12\|\partial_y w(\cdot,y)\|_2^2 - V(w(\cdot,y))$ is constant on $\mathbb{R}$. Since w_0 solves (E) we have $w \in H^2(\mathbb{R}^{N+1}) \cap C^2(\mathbb{R}^{N+1})$. Then $\|w(\cdot,y)\| \to 0$ and $\|\partial_y w(\cdot,y)\|_2 \to 0$ as $y \to \pm\infty$ and we deduce that $E_w(y) = 0$, i.e., $\frac12\|\partial_y w(\cdot,y)\|_2^2 = V(w(\cdot,y))$ for any $y \in \mathbb{R}$. Since w is even with respect to y we have $\partial_y w(\cdot,0) = 0$ and then $V(w(\cdot,0)) = 0$. Since w is radially symmetric we have $w(\cdot,0) \ne 0$ and so $w(\cdot,0) \in \mathcal{V}_+^0$. Finally, since $\partial_r w < 0$ on $\mathbb{R}^{N+1}\setminus\{0\}$ we derive that $\partial_y w(0,y) > 0$ for any $y \in (-\infty,0)$ and we conclude $V(w(\cdot,y)) = \frac12\|\partial_y w(\cdot,y)\|_2^2 > 0$ for any $y \in (-\infty,0)$.
The above results tell us that w satisfies the assumption of Lemma 3.4 on the interval $(-\infty,0)$ and $\varphi_{(-\infty,0)}(w) \ge m_0$ follows. Hence $c_{N+1} = \varphi(w) \ge 2m_0$ and we conclude that $c_{N+1} = 2m_0$.
To conclude the proof we note that since $v \ge 0$ and $-\Delta v + v = f(v) \ge 0$ on $\mathbb{R}^{N+1}$, the strong maximum principle establishes that $v > 0$ on $\mathbb{R}^{N+1}$ and so, by Theorem 1 in [18] we conclude that, up to translations, v is radially symmetric about the origin and $\partial_r v < 0$ for $r = |(x,y)| > 0$ □

References

[1] F. Alessio, *Brake orbits type solutions for a system of Allen-Cahn type equations*, Preprint (2012).

[2] F. Alessio and P. Montecchiari, *Entire solutions in $\mathbb{R}^2$ for a class of Allen-Cahn equations*, ESAIM: COCV, **11** (2005), 633–672.

[3] F. Alessio and P. Montecchiari, *Multiplicity of entire solutions for a class of almost periodic Allen-Cahn type equations*, Advanced Nonlinear Studies, **5** (2005), 515–549.

[4] F. Alessio and P. Montecchiari, *Brake orbits type solutions to some class of semilinear elliptic equations*, Calc. Var. Partial Differential Equations, **30** (2007), 51 83.

[5] A. Ambrosetti, G. Prodi *A Primer of Nonlinear Analysis*, Cambridge University Press (1995).

[6] H. Berestycki and T. Cazenave, *Instabilité des états stationnaires dans les équations de Schrödinger et de Klein-Gordon non linéaires*, C. R. Acad. Sci. Paris **293**, 9 (1981), 489–492.

[7] H. Berestycki and P.L. Lions, *Nonlinear scalar field equations, I. Existence of a ground state*, Arch. Rat. Mech. Anal. **82** (1983), 313-345.

[8] H. Berestycki, T. Gallouët and O. Kavian, *Equations de Champs scalaires euclidiens non lineaires dans le plan*, C. R. Acad. Sci; Paris Ser. I Math. **297**, 5 (1983) 307- 310

[9] J. Busca, P. Felmer, *Qualitative properties of some bounded positive solutions to scalar field equations.* Calc. Var. **13** (2001), 181-211

[10] T. Cazenave, *An introduction to nonlinear Schrödinger equations*, 26, Textos de Métodos Mathematicos, IM-UFRJ, Rio de Janeiro, (1989).

[11] T. Cazenave and P.L. Lions, *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Comm. Math. Phys., **85**, no. 4 (1982), 549-561.

[12] E. N. Dancer, *New solutions of equations on Rn, Annali Della Scuola Normale Superiore Di Pisa*, **30** (2001), 535-563.

[13] M. del Pino, M. Kowalczyk, F, Pacard and J. Wei, *The Toda system and multiple-end solutions of autonomous planar elliptic problems*, Advances in Mathematics, **224** (4-10) (2010) pp.1462-1516.

[14] C. Gui, *Hamiltonian identities for elliptic partial differential equations*, Journal of Functional Analysis **254**, no. 4 (2008), 904-933.

[15] C. Gui, A. Malchiodi and H. Xu *Axial symmetry of some steady state solutions to nonlinear Schroedinger Equations*, Proc. Amer. Math. Soc. **139**, no. 3 (2011), 1023-1032.

[16] Jeanjean L. and Tanaka K., *A remark on least energy solutions in* $\mathbb{R}^N$, Proc Amer. Math. Soc, **131** (2003), 2399–2408.

[17] M.K.Kwong, *Uniqueness of positive solutions of* $\Delta u - u + u^p$ *in* $\mathbb{R}^n$, Arch. Rational Mech. Anal. **105**, no. 3 (1989), 243–266.

[18] Y. Li and W. Ni, *Radial symmetry of positive solutions of nonlinear elliptic equations in* $\mathbb{R}^N$. Commun. Partial Differ. Equations **18**, no. 5–6 (1993), 1043-1054.

[19] P.L. Lions, *Symétrie et Compacité dans les espaces de Sobolev*, J. Funct. Anal. **49** (1982), 315-334.

[20] A. Malchiodi, *Some new entire solutions of semilinear elliptic equations on* $\mathbb{R}^n$, Adv. Math., **221**, no. 6 (2009), 1843-1909.

[21] P. Montecchiari, *Multiplicity results for a class of Semilinear Elliptic Equations on* $\mathbb{R}^m$, Rend. Sem. Mat. Univ. Padova **95** (1996), 1-36.

[22] E. H. Lieb and M. Loss, *Analysis*, Graduate Studies in Mathematics 14, American Mathematical Society, Providence, RI, (2001).

[23] L. Nirenberg, *Lectures on Linear Partial Differential Equations*, CBMS Reg. Conf. Ser. in Math., AMS, **17** (1973).

[24] L. Nirenberg, *On Elliptic partial differential equations*, Ann. Sc. Norm. Pisa, **13** (1959), 116-162.

[25] Rabinowitz, P. H., *Minimax Methods in Critical Point Theory with Applications to Differential Equations,* CBMS Regional Conference Series in Mathematics, **65**, AMS, Providence, RI, (1986).

[26] J.Serrin and M.Tang, *Uniqueness of ground states for quasilinear elliptic equations*, Indiana Univ. Math. J. **49**, no. 3 (2000), 897–923.

[27] C. Sulem and P.L. Sulem,*The nonlinear Schrödinger equation, Self-focusing and wave collapse*, Applied Mathematical Sciences, **139**, Springer-Verlag, New-York, (1999).

[28] W.A. Strauss, *Existence of solitary waves in higher dimensions*, Comm. Math. Phys., **55** (1977), 149–162.

[29] M.I. Weinstein, *Nonlinear Schrödinger equations and sharp interpolation estimates*, Comm. Math. Phys. **87**, 4 (1983), 567–576.